Math 4111
September 21, 2020 Lecture

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August 29, 2020



Figure: This is your instructor.

If $z = a + bi$ is a complex number then we define its *complex conjugate* to be the number $\overline{z} = a - bi$. We record some elementary facts about the complex conjugate:

Proposition

If z, w are complex numbers then

(1) $\overline{z + w} = \overline{z} + \overline{w}$;

(2) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$;

(3) $z + \overline{z} = 2 \cdot \mathrm{Re}\, z$;

(4) $z - \overline{z} = 2 \cdot i \cdot \mathrm{Im}\, z$;

(5) $z \cdot \overline{z} \geq 0$, *with equality holding if and only if* $z = 0$.

Proof: Write $z = a + bi, w = c + di$. Then

$$\begin{aligned}
\overline{z + w} &= \overline{(a + c) + (b + d)i} \\
&= (a + c) - (b + d)i \\
&= (a - bi) + (c - di) \\
&= \overline{z} + \overline{w}.
\end{aligned}$$

This proves **(1)**. Assertions **(2)**, **(3)**, **(4)** are proved similarly.

For **(5)**, notice that

$$z \cdot \overline{z} = (a + bi) \cdot (a - bi) = a^2 + b^2 \geq 0.$$

Clearly equality holds if and only if $a = b = 0$. □

Example

Let $z = -2 + 4i$ and $w = 5 - 3i$. Then

$$\overline{z} = -2 - 4i\,.$$

Also

$$\overline{z \cdot w} = \overline{(-2 + 4i) \cdot (5 - 3i)} = \overline{2 + 26i} = 2 - 26i$$

while

$$\overline{z} \cdot \overline{w} = (-2-4i)\cdot(5+3i) = (-10+12)+(-6-20)i = 2-26i\,. \quad \square$$

The expression $|z|$ is defined to be the nonnegative square root of $z \cdot \overline{z}$:

$$|z| = +\sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$$

when $z = x + iy$. It is called the *modulus* of z and plays the same role for the complex field that absolute value plays for the real field. It is the distance of z to the origin. The modulus has the following properties.

Proposition

If $z, w \in \mathbb{C}$ then

(1) $|z| = |\overline{z}|$;

(2) $|z \cdot w| = |z| \cdot |w|$;

(3) $|Re\, z| \leq |z| \quad , \quad |Im\, z| \leq |z|$;

(4) $|z + w| \leq |z| + |w|$;

Proof: Write $z = a + bi, w = c + di$. Then **(1)**, **(2)**, **(3)** are immediate. For **(4)** we calculate that

$$\begin{aligned}
|z+w|^2 &= (z+w)\cdot(\overline{z+w}) \\
&= z\cdot\overline{z} + z\cdot\overline{w} + w\cdot\overline{z} + w\cdot\overline{w} \\
&= |z|^2 + 2\mathrm{Re}\,(z\cdot\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\cdot\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z|\cdot|w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

Taking square roots proves **(4)**. □

Example

Let $z = 3 + 4i$ and $w = -5 + 2i$. Then

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \qquad \text{and} \qquad |w| = \sqrt{(-5)^2 + 2^2} = \sqrt{29}\,.$$

Also

$$|z \cdot w| = |(3+4i)(-5+2i)| = |-23-14i| = \sqrt{23^2 + 14^2} = \sqrt{725} = 5\sqrt{29}$$

while

$$|z| \cdot |w| = 5 \cdot \sqrt{29}\,. \qquad \square$$

Observe that, if z is real, then $z = a + 0i$ and the modulus of z equals the absolute value of a. Likewise, if $z = 0 + bi$ is pure imaginary, then the modulus of z equals the absolute value of b. In particular, the fourth part of the proposition reduces, in the real case, to the triangle inequality

$$|a + b| \leq |a| + |b| .$$

If z is any nonzero complex number, then let $r = |z|$. Now define $\xi = z/r$. We see that ξ is a complex number of modulus 1. Thus ξ lies on the unit circle, so it subtends an angle θ with the positive x-axis. Then $\xi = \cos\theta + i \sin\theta$. It is shown later in the course that

$$e^{i\theta} = \xi = \cos\theta + i \sin\theta .$$

[**Hint:** You may verify this formula for yourself by writing out the power series for the exponential and writing out the power series for cosine and sine.] We often call

$$z = re^{i\theta}$$

the *polar form* of z.

Example

Let us find all cube roots of the complex number $z = -1 + i$. Using the notation of the preceding paragraph, we see that $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. Thus $\xi = z/r = -1/\sqrt{2} + (1/\sqrt{2})i$. Examining the figure, we see that $\theta = 3\pi/4$. We have learned then that

$$z = -1 + i = \sqrt{2}e^{i3\pi/4}.$$

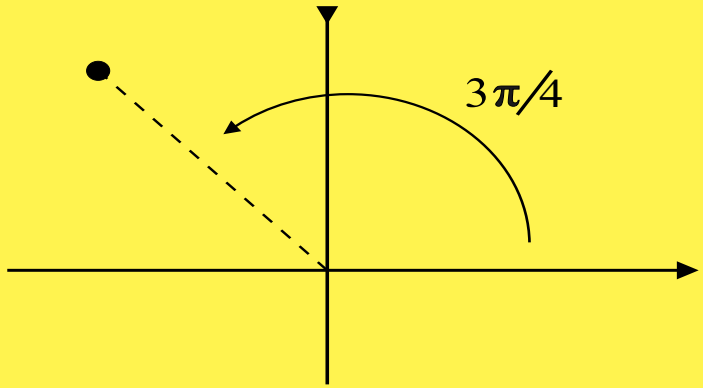


Figure: The polar form of $-1 + i$.

For the first cube root w_1 of z, we write $w_1 = se^{i\psi}$, and we solve for s and ψ. We know that

$$(w_1)^3 = z$$

so

$$\left(se^{i\psi}\right)^3 = \sqrt{2}e^{i3\pi/4}$$

or

$$s^3 e^{i3\psi} = \sqrt{2}e^{i3\pi/4}.$$

It is natural then to conclude that

$$s^3 = \sqrt{2}$$

and

$$3\psi = 3\pi/4\,.$$

We conclude that $s = 2^{1/6}$ and $\psi = \pi/4$. We have found that

$$w_1 = 2^{1/6} e^{i\pi/4}$$

is a cube root of z. But this is not the only cube root! There are three cube roots in total.

We next notice that z can also be written

$$z = \sqrt{2}e^{i((3\pi/4)+2\pi)}.$$

[Observe that there is some ambiguity built into the polar form of a complex number, just as there is ambiguity in the polar coordinates that you learned about in calculus. The reason is that the cosine and sine functions are 2π-periodic.]

Now we repeat the calculation above with this new form for the complex number z. We know that

$$(w_2)^3 = z$$

so

$$\left(se^{i\psi}\right)^3 = \sqrt{2}e^{i((3\pi/4)+2\pi)}$$

or

$$s^3 e^{i3\psi} = \sqrt{2}e^{i11\pi/4}\,.$$

It is natural then to conclude that

$$s^3 = \sqrt{2}$$

and

$$3\psi = 11\pi/4\,.$$

We conclude that $s = 2^{1/6}$ and $\psi = 11\pi/12$. We have found that

$$w_2 = 2^{1/6} e^{i11\pi/12}$$

is a cube root of z.

Let us do the calculation one more time with z now written as

$$z = \sqrt{2}e^{i((3\pi/4)+4\pi)}$$

(again we exploit the periodicity of sine and cosine). We know that

$$(w_3)^3 = z$$

so

$$\left(se^{i\psi}\right)^3 = \sqrt{2}e^{i((3\pi/4)+4\pi)}$$

or

$$s^3 e^{i3\psi} = \sqrt{2}e^{i19\pi/4}.$$

It is natural then to conclude that

$$s^3 = \sqrt{2}$$

and

$$3\psi = 19\pi/4\,.$$

We conclude that $s = 2^{1/6}$ and $\psi = 19\pi/12$. We have found that

$$w_3 = 2^{1/6} e^{i19\pi/12}$$

is a cube root of z.

There is no sense to repeat these calculations any further. It is true that $z = \sqrt{2}e^{i(3\pi/4+6\pi)}$. But performing our calculations for this form of z would simply cause us to rediscover w_1. We have found three cube roots of z—which is the right number—and that is the end of the calculation. □

We conclude this discussion by recording the most important basic fact about the complex numbers. Carl Friedrich Gauss gave five proofs of this theorem (the Fundamental Theorem of Algebra) in his doctoral dissertation:

Theorem

Let $p(z)$ be any polynomial of degree at least 1. Then p has a root $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Using a little algebra, one can in fact show that a polynomial of degree k has k roots (counting multiplicity).