Math 4111
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Figure: This is your instructor.

When we first learn about the complex numbers, the most troublesome point is the very beginning: "Let's pretend that the number -1 has a square root. Call it i." What gives us the right to "pretend" in this fashion? The answer is that we have no such right.[1] If -1 has a square root, then we should be able to construct a number system in which that is the case. That is what we shall do in this section.

[1] The complex numbers were initially developed so that we would have a number system in which all polynomial equations are solvable. One of the reasons, historically, that mathematicians had trouble accepting the complex numbers is that they did not believe that they really existed—they were just made up. This is, in part, how they came to be called "imaginary" and "complex." Mathematicians had similar trouble accepting negative numbers; for a time, negative numbers were called "forbidden."

Definition

The system of *complex numbers*, denoted by the symbol $\mathbb{C}$, consists of all ordered pairs (a, b) of real numbers. We add two complex numbers (a, b) and $(\widetilde{a}, \widetilde{b})$ by the formula

$$(a, b) + (\widetilde{a}, \widetilde{b}) = (a + \widetilde{a}, b + \widetilde{b}) .$$

We multiply two complex numbers by the formula

$$(a, b) \cdot (\widetilde{a}, \widetilde{b}) = (a \cdot \widetilde{a} - b \cdot \widetilde{b}, a \cdot \widetilde{b} + \widetilde{a} \cdot b) .$$

Remark

If you are puzzled by this definition of multiplication, do not worry. In a few moments you will see that it gives rise to the notion of multiplication of complex numbers that you are accustomed to. Perhaps more importantly, a naive rule for multiplication like $(a, b) \cdot (\widetilde{a}, \widetilde{b}) = (a\widetilde{a}, b\widetilde{b})$ gives rise to nonsense like $(1, 0) \cdot (0, 1) = (0, 0)$. It is really necessary for us to use the initially counterintuitive definition of multiplication that is presented here.

Example

Let $z = (3, -2)$ and $w = (4, 7)$ be two complex numbers. Then

$$z + w = (3, -2) + (4, 7) = (3 + 4, -2 + 7) = (7, 5)\,.$$

Also

$$z \cdot w = (3, -2) \cdot (4, 7) = (3 \cdot 4 - (-2) \cdot 7, 3 \cdot 7 + 4 \cdot (-2)) = (26, 13)\,. \quad \square$$

As usual, we ought to check that addition and multiplication are commutative, associative, that multiplication distributes over addition, and so forth. We shall leave these tasks to the exercises. Instead we develop some of the crucial, and more interesting, properties of our new number system.

Theorem

The following properties hold for the number system $\mathbb{C}$.

(a) *The number $1 \equiv (1,0)$ is the multiplicative identity: $1 \cdot z = z \cdot 1 = z$ for any $z \in \mathbb{C}$.*

(b) *The number $0 \equiv (0,0)$ is the additive identity: $0 + z = z + 0 = z$ for any $z \in \mathbb{C}$.*

(c) *Each complex number $z = (x,y)$ has an additive inverse $-z = (-x,-y)$: it holds that $z + (-z) = (-z) + z = 0$.*

(d) *The number $i \equiv (0,1)$ satisfies $i \cdot i = -1$; in other words, i is a square root of -1.*

Proof: These are direct calculations, but it is important for us to work out these facts.

First, let $z = (x, y)$ be any complex number. Then

$$1 \cdot z = (1, 0) \cdot (x, y) = (1 \cdot x - 0 \cdot y, 1 \cdot y + x \cdot 0) = (x, y) = z\,.$$

This proves the first assertion.

For the second, we have

$$0 + z = (0, 0) + (x, y) = (0 + x, 0 + y) = (x, y) = z\,.$$

With z as above, set $-z = (-x, -y)$. Then

$$z + (-z) = (x, y) + (-x, -y) = (x + (-x), y + (-y)) = (0, 0) = 0\,.$$

Finally, we calculate

$$i \cdot i = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 0 \cdot 1) = (-1,0) = -1\,.$$

Thus, as asserted, i is a square root of -1. □

Proposition

If $z \in \mathbb{C}, z \neq 0$, then there is a complex number w such that $z \cdot w = 1$.

Proof: You might be thinking, "Well, of course $w = 1/z$. But this is nonsense. The expression $1/z$ is *not* written in the form of a complex number!

Write $z = (x, y)$ and set

$$w = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

Since $z \neq 0$, $x^2 + y^2 \neq 0$, so this definition makes sense. Then it is straightforward to verify that $z \cdot w = 1$:

$$\begin{aligned} z \cdot w &= (x, y) \cdot \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \\ &= \left(x \cdot \frac{x}{x^2 + y^2} - y \cdot \frac{-y}{x^2 + y^2}, x \cdot \frac{-y}{x^2 + y^2} + \frac{x}{x^2 + y^2} \cdot y \right) \\ &= \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{-yx + xy}{x^2 + y^2} \right) \\ &= (1, 0) \\ &= 1 . \end{aligned}$$



Example

Consider the complex number $z = 2 + 3i$. According to the proposition,

$$w = \left(\frac{2}{2^2 + 3^2}, \frac{-3}{2^2 + 3^2} \right) = \left(\frac{2}{13}, \frac{-3}{13} \right)$$

will be the multiplicative inverse of z. And, indeed,

$$\begin{aligned} z \cdot w &= (2, 3) \cdot \left(\frac{2}{13}, \frac{-3}{13} \right) \\ &= \left(\frac{4}{13} + \frac{9}{13}, \frac{-6}{13} + \frac{6}{13} \right) \\ &= \left(\frac{13}{13}, 0 \right) = (1, 0) = 1 \,. \end{aligned}$$

□

Of course we interpret the quotient z/w of complex numbers to mean $z \cdot (1/w)$. This will be a new complex number.

Thus every nonzero complex number has a multiplicative inverse. The other field axioms for $\mathbb{C}$ are easy to check. We conclude that the number system $\mathbb{C}$ forms a field. You will prove in the exercises that it is not possible to order this field. If α is a real number then we associate α with the complex number $(\alpha, 0)$. Thus we have the natural "embedding"

$$\mathbb{R} \ni \alpha \longmapsto (\alpha, 0) \in \mathbb{C}\,.$$

In this way, we can think of the real numbers as a *subset* of the complex numbers. In fact, the real field $\mathbb{R}$ is a *subfield* of the complex field $\mathbb{C}$. This means that if $\alpha, \beta \in \mathbb{R}$ and $(\alpha, 0), (\beta, 0)$ are the corresponding elements in $\mathbb{C}$ then $\alpha + \beta$ corresponds to $(\alpha + \beta, 0)$ and $\alpha \cdot \beta$ corresponds to $(\alpha \cdot \beta, 0)$. These assertions are explored more thoroughly in the exercises.

With the remarks in the preceding paragraph we can sometimes ignore the distinction between the real numbers and the complex numbers. For example, we can write

$$5 \cdot i$$

and understand that it means $(5,0) \cdot (0,1) = (0,5)$. Likewise, the expression

$$5 \cdot 1$$

can be interpreted as $5 \cdot 1 = 5$ or as $(5,0) \cdot (1,0) = (5,0)$ without any danger of ambiguity.

Theorem

Every complex number can be written in the form $a + b \cdot i$, where a and b are real numbers. In fact, if $z = (x, y) \in \mathbb{C}$ then

$$z = x + y \cdot i \, .$$

Proof: With the identification of real numbers as a subfield of the complex numbers, we have that

$$x + y \cdot i = (x, 0) + (y, 0) \cdot (0, 1) = (x, 0) + (0, y) = (x, y) = z$$

as claimed. □

Now that we have constructed the complex number field, we will adhere to the usual custom of writing complex numbers as $z = a + b \cdot i$ or, more simply, $a + bi$. We call a the *real part* of z, denoted by $\operatorname{Re} z$, and b the *imaginary part* of z, denoted $\operatorname{Im} z$. We have

$$(a + bi) + (\widetilde{a} + \widetilde{b}i) = (a + \widetilde{a}) + (b + \widetilde{b})i$$

and

$$(a + bi) \cdot (\widetilde{a} + \widetilde{b}i) = (a \cdot \widetilde{a} - b \cdot \widetilde{b}) + (a \cdot \widetilde{b} + \widetilde{a} \cdot b)i\,.$$

Example

Let $z = 3 - 7i$ and $w = 4 + 6i$. Then

$$z + w = (3 - 7i) + (4 + 6i) = 7 - i\,,$$

$$\begin{aligned} z \cdot w &= (3 - 7i) \cdot (4 + 6i) \\ &= (3 \cdot 4 - (-7) \cdot 6) + (3 \cdot 6 + (-7) \cdot 4)i \\ &= 54 - 10i\,. \end{aligned}$$

□