Math 4111
September 14, 2020 Lecture

Steven G. Krantz

August 29, 2020



Figure: This is your instructor.

This is a book about analysis in the real number system. Such a study must be founded on a careful consideration of *what the real numbers are* and *how they are constructed*. In the present lecture we give a careful treatment of the real number system. In the next we consider the complex numbers.

We know from calculus that, for many purposes, the rational numbers are inadequate. It is important to work in a number system which is closed with respect to the operations we shall perform. This includes limiting operations. While the rationals are closed under the usual arithmetic operations (addition, subtraction, multiplication, division), they are *not* closed under the mathematical operation of taking *limits*. For instance, the sequence of rational numbers $3, 3.1, 3.14, 3.141, \ldots$ consists of terms that seem to be getting closer and closer together, *seem* to tend to some limit, and yet there is no rational number which will serve as a limit (of course it turns out that the limit is π—an "irrational" number).

We will now deal with the real number system, a system which contains all limits of sequences of rational numbers (as well as all limits of sequences of real numbers!). In fact our plan will be as follows: in this section we shall discuss all the requisite properties of the reals. The actual construction of the reals is rather subtle, and we shall put that in an Appendix to Section 1.1.

Definition

Let A be an ordered set and X a subset of A. The set X is called *bounded above* if there is an element $b \in A$ such that $x \leq b$ for all $x \in X$. We call the element b an *upper bound* for the set X.

Example

Let $A = \mathbb{Q}$ (the rational numbers) with the usual ordering. The set $X = \{x \in \mathbb{Q} : 2 < x < 4\}$ is bounded above. For example, 15 is an upper bound for X. So are the numbers 12 and 4. It is interesting to observe that no element of this particular X can actually be an upper bound for X. The number 4 is a good candidate, but 4 is not an element of X. In fact if $b \in X$ then $(b+4)/2 \in X$ and $b < (b+4)/2$, so b could not be an upper bound for X. □

It turns out that the most convenient way to formulate the notion that the real numbers have "no holes" (i.e. that all sequences which seem to be converging actually have something to converge to) is in terms of upper bounds.

Definition

Let A be an ordered set and X a subset of A. An element $b \in A$ is called a *least upper bound* (or *supremum*) for X if b is an upper bound for X and $b \leq b'$ for every upper bound b' for X. We denote the supremum of X by $\sup X$. The supremum is also sometimes called the *least upper bound* and denoted by $\operatorname{lub} X$.

By its very definition, if a least upper bound exists then it is unique. Notice that we *could have* phrased the definition as "The point b is the least upper bound for X if, whenever $c < b$, then c cannot be an upper bound for X."

Example

In the last example, we considered the set X of rational numbers strictly between 2 and 4. We observed there that 4 is the least upper bound for X. Note that this least upper bound is not an element of the set X.

The set $Y = \{y \in \mathbb{Z} : -9 \leq y \leq 7\}$ has least upper bound 7. In this case, the least upper bound *is* an element of the set Y. □

Notice that we may define a lower bound for a subset of an ordered set in a fashion similar to that for an upper bound:

Definition

A point $\ell \in A$ is a lower bound for $X \subseteq A$ if $\ell \leq x$ for all $x \in X$. A *greatest lower bound* (or *infimum*) for X is then defined to be a lower bound c such that $c \geq c'$ for every lower bound c' for X. We denote the infimum of X by $\inf X$. The infimum is also sometimes called the *greatest lower bound* and denoted by $\operatorname{glb} X$.

As with the least upper bound, we may note that the definition of greatest lower bound could be phrased in this way: "The point c is the greatest lower bound for X if, whenever $e > c$, then e cannot be a lower bound for X."

Example

The set $X = \{x \in \mathbb{Q} : 2 < x < 4\}$ in the last two examples has lower bounds -20, 0, 1, 2, for instance. The greatest lower bound is 2, which is *not* an element of the set.

The set $Y = \{y \in \mathbb{Z} : -9 \leq y \leq 7\}$ in the last example has lower bounds—among others—given by $-53, -22, -10, -9$. The number -9 is the greatest lower bound. It *is* an element of Y. □

The purpose that the real numbers will serve for us is as follows: they will contain the rationals, they will still be an ordered field (a field is a set with arithmetic operations $+$ and $\cdot$ —see the Appendix at the end of Section 1.1), and *every subset which has an upper bound will have a least upper bound*. [See [KRA1] for a thorough treatment of the concept of ordered field.] We formulate this result as a theorem.

Theorem

There exists an ordered field $\mathbb{R}$ which **(i)** *contains $\mathbb{Q}$ and* **(ii)** *has the property that any nonempty subset of $\mathbb{R}$ which has an upper bound has a least upper bound (in the number system $\mathbb{R}$).*

The last property described in this theorem is called the Least Upper Bound Property of the real numbers. As mentioned previously, this theorem will be proved in the Appendix to Section 1.1. Of course the least upper bound property, in and of itself, is something of a technicality. But we shall see that a great many interesting and powerful properties of the real numbers can be derived from it.

Now we begin to realize why it is so important to *construct* the number systems that we will use. We are endowing $\mathbb{R}$ with a great many properties. Why do we have any right to suppose that there exists a set with all these properties? We must produce one! We do so in the Appendix.

Let us begin to explore the richness of the real numbers. The next theorem states a property which is not shared by the rationals. It is fundamental in its importance.

Theorem

Let x be a positive real number. Then there is a positive real number y such that $y^2 = y \cdot y = x$.

Proof: We will use throughout this proof the fact that, if $0 < a < b$ then $a^2 < b^2$.

Let

$$S = \{s \in \mathbb{R} : s > 0 \text{ and } s^2 < x\}.$$

Then S is not empty since $x/2 \in S$ if $x < 2$ and $1 \in S$ otherwise. Also S is bounded above since $x + 1$ is an upper bound for S. By our main theorem, the set S has a least upper bound. Call it y. Obviously, $0 < \min\{x/2, 1\} \leq y$ hence y is positive. We claim that $y^2 = x$. To see this, we eliminate the other two possibilities.

If $y^2 < x$ then set $\epsilon = (x - y^2)/[4(x+1)]$. Then $\epsilon > 0$ and

$$\begin{aligned}
(y+\epsilon)^2 &= y^2 + 2\cdot y\cdot \epsilon + \epsilon^2 \\
&= y^2 + 2\cdot y\cdot \frac{x-y^2}{4(x+1)} + \frac{x-y^2}{4(x+1)}\cdot\frac{x-y^2}{4(x+1)} \\
&< y^2 + 2\cdot y\cdot \frac{x-y^2}{4y} + \frac{x-y^2}{4(x+1)}\cdot\frac{x-y^2}{4(x+1)} \\
&< y^2 + \frac{x-y^2}{2} + \frac{x-y^2}{4}\cdot\frac{x}{4x} \\
&< y^2 + (x-y^2) \\
&= x.
\end{aligned}$$

Thus $y+\epsilon \in S$, and y cannot be an upper bound for S. This contradiction tells us that $y^2 \not< x$.

Similarly, if it were the case that $y^2 > x$ then we set $\epsilon = (y^2 - x)/[4(x+1)]$. A calculation like the one we just did then shows that $(y-\epsilon)^2 \geq x$. Hence $y - \epsilon$ is also an upper bound for S, and y is therefore not the *least* upper bound. This contradiction shows that $y^2 \not> x$.

The only remaining possibility is that $y^2 = x$. □

Remark

The theorem tells us in particular that $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{11}$, etc. all exist in the real number system. And each of these numbers is irrational (see an upcoming theorem where it is shown that $\sqrt{2}$ is irrational). In fact the only square roots of integers that are *not* irrational are the square roots of the perfect squares $1, 4, 9, 16, 25, \ldots$. □

A similar proof shows that, if n is a positive integer and x a positive real number, then there is a positive real number y such that $y^n = x$.

We next use the Least Upper Bound Property of the Real Numbers to establish two important qualitative properties of the Real Numbers:

Theorem

The set $\mathbb{R}$ of real numbers satisfies the Archimedean Property:

Let a and b be positive real numbers. Then there is a natural number n such that $na > b$.

The set $\mathbb{Q}$ of rational numbers satisfies the following Density Property:

Let $c < d$ be real numbers. Then there is a rational number q with $c < q < d$.

Proof: Suppose the Archimedean Property to be false. Then $S = \{na : n \in \mathbb{N}\}$ has b as an upper bound. Therefore S has a finite supremum β. Since $a > 0$, it follows that $\beta - a < \beta$. So $\beta - a$ is not an upper bound for S, and there must be a natural number n' such that $n' \cdot a > \beta - a$. But then $(n' + 1)a > \beta$, and β cannot be the supremum for S. This contradiction proves the first assertion.

For the second property, let $\lambda = d - c > 0$. By the Archimedean Property, choose a positive integer N such that $N \cdot \lambda > 1$. Again the Archimedean Property gives a natural number P such that $P > N \cdot c$ and another Q such that $Q > -N \cdot c$. Thus we see that Nc falls between the integers $-Q$ and P; therefore there must be an integer M between $-Q$ and P such that

$$M - 1 \leq Nc < M\,.$$

Thus $c < M/N$. Also

$$M \leq Nc + 1 \;\text{ hence }\; \frac{M}{N} \leq c + \frac{1}{N} < c + \lambda = d.$$

So M/N is a rational number lying between c and d. $\square$

Remark

The density property above says that, between any two real numbers is a rational number. Even more can be said. In fact **(i)** between every two irrational numbers is a rational number and **(ii)** between every two rational numbers is an irrational number.

In Appendix II later on we establish that the set of all decimal representations of numbers is uncountable. It follows that the set of all real numbers is uncountable. In fact the same proof shows that the set of all real numbers in the interval $(0, 1)$, or in any nonempty open interval (c, d), is uncountable.

The set $\mathbb{R}$ of real numbers is uncountable, yet the set $\mathbb{Q}$ of rational numbers is countable. It follows that the set $\mathbb{R} \setminus \mathbb{Q}$ of *irrational* numbers is uncountable. In particular, it is nonempty. Thus we may see with very little effort that there exist a great many real numbers which cannot be expressed as a quotient of integers. However, it can be quite difficult to see whether any particular real number (such as π or e or $\sqrt[5]{2}$) is irrational.

We conclude by recalling the "absolute value" notation:

Definition

Let x be a real number. We define

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

It is left as an exercise for you to verify the important *triangle inequality*:

$$|x + y| \leq |x| + |y| .$$

[**Hint:** It is convenient to verify that the square of the left-hand side is less than or equal to the square of the right-hand side.]