Math 310
Krantz

SOLUTIONS

PRACTICE EXAM FOR SECOND MIDTERM

(8 points) **1.** Let $S = \{3, 4, 6\}$ and $T = \{3, 5\}$. What is $S \cup T$? What is $S \cap T$? What is $S \setminus T$?

$S \cup T = \{3, 4, 5, 6\}$

$S \cap T = \{3\}$

$S \setminus T = \{4, 6\}$

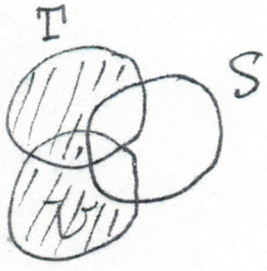
(8 points) **2.** Let $S = \{2, 4\}$ and $T = \{a, b, d\}$. What is $S \times T$? What is $T \times S$?

$S \times T = \{(2,a),(2,b),(2,d),(4,a),(4,b),(4,d)\}$

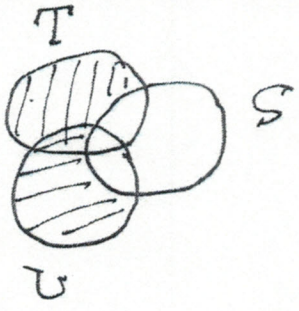
$T \times S = \{(a,2),(a,4),(b,2),(b,4),(d,2),(d,4)\}$

(8 points) **3.** Draw two Venn diagrams to illustrate the identity

$$(T \cup U) \setminus S = (T \setminus S) \cup (U \setminus S).$$



$(T \cup U) \setminus S$



$(T \setminus S) \cup (U \setminus S)$

(10 points) **4.** What is the power set of $\{\lambda, A, 2\}$?

$\{\{\lambda\}, \{A\}, \{2\}, \{\lambda, A\}, \{\lambda, 2\}, \{A, 2\}, \{\lambda, A, 2\}, \phi\}$

(12 points) **5.** Which of these functions is one-to-one? Which is onto (give a brief reason for each answer)?

(a) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 + x$

$f(0) = f(-1) = 0$ so not one-to-one

quadratic formula $\Rightarrow \nexists x$ with $f(x) = -5$, not onto.

(b) $g : \mathbb{N} \to \mathbb{N}$ $\quad g(n) = n(n+3)$

g is increasing so one-to-one

$\nexists n$ with $g(n) = 1$

(c) $h : \mathbb{R} \to \mathbb{R}$ $\quad h(x) = x \cos x$

h is continuous & takes arbitrarily large positive values and arbitrarily large negative values. So is onto.

A horizontal line will intersect the graph more than once so <u>not</u> one-to one.

(12 points) **6.** Which of these sets is countable and which uncountable (give a brief reason for each answer)?

(a) $\mathbb{C}\times\mathbb{C}$

$f : \mathbb{C} \to \mathbb{C}\times\mathbb{C}$

$z \mapsto (z, 0)$ is one-to-one,

So $\text{card}(\mathbb{C}) \leq \text{card}(\mathbb{C}\times\mathbb{C})$, Hence $\mathbb{C}\times\mathbb{C}$ uncountable.

(b) $\mathbb{Z}\times\mathbb{Q}$

$\mathbb{Z}$ is countable, $\mathbb{Q}$ is countable,

So $\mathbb{Z}\times\mathbb{Q}$ is countable

(c) $\mathbb{N}\times\mathbb{R}$

$g : \mathbb{R} \to \mathbb{N}\times\mathbb{R}$

$x \mapsto (0, x)$ is one-to-one,

So $\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{N}\times\mathbb{R})$,

Hence $\mathbb{N}\times\mathbb{R}$ uncountable.

(10 points) **7.** Calculate the inverse of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0. \end{cases}$$

$$f^{-1}(x) = \begin{cases} x^{1/3}, & x \leq 0 \\ x, & x > 0 \end{cases}$$

(14 points) **8.** Prove that the collection of S of rational numbers with denominator 7 is countable.

$f : S \to \mathbb{Z}$

$\frac{m}{7} \to m$

is one-to-one and onto. So

S is countable.

9. Explain why the product of an uncountable set and a countable set is uncountable.

Let S be countable and T uncountable.

Let $s_0 \in S$. The function

$$f : T \to S \times T$$
$$t \mapsto (s_0, t)$$

is one-to-one, so

$$\text{card}(T) \leq \text{card}(S \times T).$$

Since T is uncountable, it follows that $S \times T$ is uncountable.

10. Explain why the union of a uncountable set and a countable set is uncountable.

Let S be countable and T uncountable.

The function

$$g : T \to S \cup T$$
$$t \mapsto t$$

is one to one. So

$$\text{card}(T) \leq \text{card}(S \cup T).$$

Since T is uncountable, so

$S \cup T$ is uncountable.

11. Prove that subtraction in the integers is well defined. You should use the actual, *rigorous* definition of the integers in terms of ordered pairs of natural numbers to do this problem. Also use the rigorous definition of subtraction provided in the book and the lectures.

Let $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$.
So $(a+b' = a'+b)$ * and $(c+d' = c'+d)$ ** Now
$[(a,b)] - [(c,d)] = [(a+d, b+c)]$ and $[(a',b') - (c',d')] = [(a'+d', c'+b')]$
We want to see that these two answers are the same.
Add the left side of * to the right side of **
and the right side of * to the left side of **.
We get.
$a+b'+c'+d = a'+b+c+d'$ or
$(a+d)+(b'+c') = (a'+d')+(b+c)$.
But this says that $(a+d, b+c) \sim (a'+d', b'+c')$,
which is what we want.

12. What is the multiplicative inverse of the complex number $-i$?

It is i.

13. Find a square root in the quaternions of the quaterion $2 \cdot 1 + 2 \cdot \mathbf{j}$.

Guess a quaternion of the form $\alpha \cdot \underline{\underline{1}} + \beta \underline{\underline{j}}$.

Then $(\alpha \cdot \underline{\underline{1}} + \beta \underline{\underline{j}})(\alpha \underline{\underline{1}} + \beta \underline{\underline{j}}) = \alpha^2 \underline{\underline{1}} - \beta^2 \underline{\underline{1}} + 2\alpha\beta \underline{\underline{j}}$

So $(\alpha^2 - \beta^2)\,\mathbb{1} + 2\alpha\beta \underline{\underline{j}} = 2\underline{\underline{i}} + 2\underline{\underline{j}}$.

$\alpha^2 - \beta^2 = 2$

$2\alpha\beta = 2 \Rightarrow \alpha = \frac{1}{\beta}$

$\frac{1}{\beta^2} - \beta^2 = 2 \Rightarrow \beta^4 + 2\beta^2 - 1 = 0$

$\beta^2 = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$, $\alpha^2 = \frac{1}{-1 \pm \sqrt{2}}$

$\beta = \sqrt{\sqrt{2} - 1}$, $\alpha = \frac{1}{\sqrt{\sqrt{2} - 1}}$.

Sol. is $\frac{1}{\sqrt{\sqrt{2}-1}} \underline{\underline{1}} + \sqrt{\sqrt{2} - 1}\, \underline{\underline{j}}$

14. Find all square roots of the complex number $1 - i$.

$1 - i = \sqrt{2}\, e^{\frac{7\pi}{4} i}$

Solve $r^2 e^{2i\theta} = \sqrt{2}\, e^{\frac{7\pi}{4} i}$

so $r = 2^{1/4}$, $\theta = \frac{7\pi}{8}$

first root is $z_1 = 2^{1/4} e^{i\frac{7\pi}{8}}$

Now solve $r^2 e^{2i\theta} = \sqrt{2}\, e^{\frac{15\pi}{4} i}$

so $r = 2^{1/4}$, $\theta = \frac{15\pi}{8}$

second root is $z_2 = 2^{1/4} e^{i\frac{15\pi}{8}}$