name MATH 310

date

project SOLUTIONS TO CROWDMARK 7

tasks Chapter 7

1. Let $\rho > 0$ be small. Create a Cantor-like set by first removing from $[0,1]$ an open interval of length ρ. Then remove two open intervals of length ρ^2. Then remove four intervals of length ρ^3. And so forth.

The total length of all the removed intervals is

$$\rho + 2\rho^2 + 4\rho^3 + \cdots = \sum_{j=1}^{\infty} 2^{j-1}\rho^j$$

$$= \sum_{j=0}^{\infty} 2^j \rho^{j+1} = \rho \sum_{j=0}^{\infty} (2\rho)^j = \frac{\rho}{1-2\rho}.$$

Set $\eta = \frac{\rho}{1-2\rho}$

so $\rho = \frac{\eta}{1+2\eta}$. (*)

So if we run this paradigm with the value of ρ in (*) then we get a Cantor set whose complement in $[0,1]$ has length η.

2. The Cantor set has length 0. So the interval (x_1, x_2) cannot all be elements of the Cantor set. So there is a t, $x_1 < t < x_2$, s.t. t is not in the Cantor set.

name

date

project

tasks

3. We construct the Cantor set by removing countably many intervals, and each interval has two endpoints, So the number of endpoints is countable.

9. $\mathbb{Q}$ is not discrete because if $q \in \mathbb{Q}$ and $\epsilon > 0$ is small then $(q-\epsilon, q+\epsilon)$ will contain infinitely many elements of $\mathbb{Q}$. The Cantor set is uncountable so cannot be discrete. $\mathbb{Z}$ is discrete. If $n \in \mathbb{Z}$, then $(n-\frac{1}{2}, n+\frac{1}{2}) \cap \mathbb{Z} = \{n\}$. T is discrete because if $\frac{1}{j} \in T$ then

$$\left(\frac{1}{j} - \frac{1}{j^3}, \frac{1}{j} + \frac{1}{j^3}\right) \cap T = \left\{\frac{1}{j}\right\}.$$

11. If the sequence is bounded above then let α be its least upper bound. If $\epsilon > 0$ then $\exists J$ such that $|s_J - \alpha| < \epsilon$. But then $|s_j - \alpha| < \epsilon \ \forall j > J$. So $s_j \to \alpha$.

Otherwise the sequence is unbounded.

16. Define a relation on U by $x \sim y$ if all points between x and y lie in U. This is an equivalence relation. The equivalence classes are pairwise disjoint, and they are intervals.

name

date

project

3

tasks

29. Let $\{x_j\}$ be an enumeration of the rational numbers. Then the closure of $\{x_j\}$ is $\mathbb{R}$. And $\mathbb{R} \setminus \{x_j\}$ is uncountable.

22. Let $\mathcal{O}_j$ be open for each j. Let $\mathcal{O} = \bigcup \mathcal{O}_j$. If $x \in \mathcal{O}$, then $x \in \mathcal{O}_j$ for some j. Then $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \mathcal{O}_j \subset \mathcal{O}$. So $\mathcal{O}$ is open.

If $\mathcal{O}_j = \left(-\frac{1}{j}, \frac{1}{j}\right)$ then each $\mathcal{O}_j$ is open but $\bigcap_j \mathcal{O}_j = \{0\}$ which is not open.

Let $E_j = \left[\frac{1}{j}, 1 - \frac{1}{j}\right]$. Then each E_j is closed. But

$$\bigcup_j E_j = (0, 1)$$

which is not closed.

Let E_j be closed and $E = \bigcap_j E_j$. Then

$$^cE = {}^c\left(\bigcap_j E_j\right) = \bigcup_j {}^cE_j.$$

Each ${}^cE_j$ is open so, by the above, $\bigcup_j E_j$ is open. Hence E is closed.

Chapter 8

2. The union of sets of irrational numbers is a set of irrational numbers. The finite intersection of sets of irrational numbers is a set of irrational numbers. So this is a topology.

name

date

project

4

tasks

4. Let $U_j = \{j\}$. Then

$$\bigcup_{j=2}^{\infty} U_j = \mathbb{N},$$ which is not finite.

So the collection of finite subsets of $\mathbb{R}$ is not a topology.

The collection of infinite subsets of $\mathbb{Z}$ is a topology.

17. Let $f:(0,1) \to (0,1)$ be given by $f(x) = 1-x^2$. Then

$$f^{-1}\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right) = (-1, -\tfrac{1}{\sqrt{2}}] \cup [\tfrac{1}{\sqrt{2}}, 1)$$

which is not compact.

21. The closure of S is just S itself. There are no limit points outside S.
The interior of S is empty. The boundary of S is S.