MATH 310

FINAL EXAM SOLUTIONS

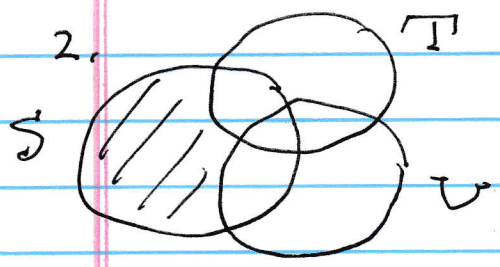
1.

A	B	~A	~B	A∨B	~(A∨B)	~A∧~B	(~A∧~B)⇒(A∨B)
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	F

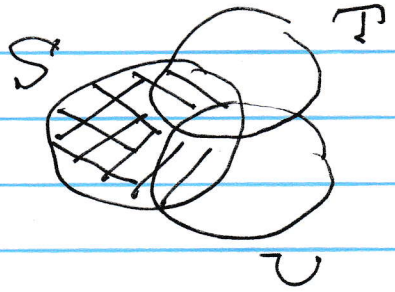
~(A∨B)⇒(A∨B)
T
T
T
F

These are the same, so logically equivalent.

2.



$S \setminus (T \cup U)$



$(S \setminus T) \cap (S \setminus U)$

3. (a) $f(0) = f(-1)$ so not one-to-one.
The graph is an upward-opening parabola, so not onto.

(b) $g(0) = g(1) = 0$ so not one to one.
The quadratic formula shows $\nexists$ n such that $g(n) = -5$. So not onto.

(c) $h(0) = h(\frac{\pi}{2}) = 0$ so not one-to-one.
h is continuous & takes arbitrarily large positive values and arbitrarily large negative values, So, by intermediate value property, h is onto.

4. a) Let $X = \{iy : y \in \mathbb{R}\}$. Then X is uncountable.
And

$$\Phi : X \to \mathbb{C} \setminus \mathbb{R}$$
$$iy \to 0 + iy$$

is one-to-one. So

$$\text{card}(X) \leq \text{card}(\mathbb{C} \setminus \mathbb{R}).$$

Hence $\mathbb{C} \setminus \mathbb{R}$ is uncountable.

b) Let $\Phi : \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$
$x \mapsto (0, x)$.
Then Φ is one-to-one. So

$$\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{Z} \times \mathbb{R}),$$

We conclude that $\mathbb{Z} \times \mathbb{R}$ is uncountable.

c) The product of two countable sets is countable.
So $\mathbb{Z} \times \mathbb{N}$ is countable.

5. If n^2 is a pefect square then the next pefect squre is $(n+1)^2 = n^2+2n+1$. And $(n^2+2n+1) - n^2 = 2n+1 > 1$. So pefect squres never occur in succession.

6. Let $P(n)$ be $1^2+2^2+\cdots+n^2 = \frac{2n^3+3n^2+n}{6}$.

$P(1)$ is $1^2 = \frac{2\cdot 1^3+3\cdot 1^2+1}{6} = 1$ so true

Now assume $P(n)$. Add $(n+1)^2$ to both sides.

Thus

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{2n^3+3n^2+n}{6} + (n+1)^2$$

$$= \frac{2n^3+3n^2+n}{6} + (n^2+2n+1)$$

$$= \frac{2n^3+9n^2+13n+6}{6}$$

$$= \frac{2(n+1)^3+3(n+1)^2+(n+1)}{6}$$

and this is $P(n+1)$. The induction is complete.

7. Begin with the unit interval $[0, 1]$. At the first step remove an interval of length $2/7$. At the second step remove two intervals of length $(2/7)^2$. At the third step remove four intervals of length $(2/7)^3$. Etcetera.

The total length of all intervals removed is

$$\sum_{j=1}^{\infty} 2^{j-1}\left(\frac{2}{7}\right)^j = \frac{2}{7}\sum_{j=1}^{\infty}\left(\frac{2}{7}\right)^{j-1}\cdot 2^{j-1}$$

$$= \frac{2}{7}\sum_{j=1}^{\infty}\left(\frac{4}{7}\right)^{j-1} = \frac{2}{7}\sum_{j=0}^{\infty}\left(\frac{4}{7}\right)^{j} = \frac{2}{7}\cdot\frac{1}{1-4/7}$$

$$= \frac{2}{7-4} = \frac{2}{3}.$$

So our Cantor-like set has length $1/3$.

8. Assume to the contrary that 6 has a rational square root $\frac{a}{b}$ in lowest terms. Then

$$6 = \left(\frac{a}{b}\right)^2$$

$$6b^2 = a^2 \qquad (*)$$

Then 2 divides the lefthand side so 2 divides the righthand side. Thus 2 divides a. Write $a = 2\alpha$, some $\alpha \in \mathbb{N}$. Then $(*)$ says that

$$6b^2 = (2\alpha)^2 = 4\alpha^2$$

$$3b^2 = 2\alpha^2.$$

We see that 2 divides the righthand side so 2 divides the lefthand side. So 2 divides b.

We have shown that a and b have a common factor of 2. Contradiction. So 6 does <u>not</u> have a rational square root.

9. Suppose that $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$. Then $a \cdot b' = a' \cdot b$ and $c \cdot d' = c' \cdot d$. Multiplying together left sides and right sides gives

$$a \cdot b' \cdot c \cdot d' = a \cdot b \cdot c' \cdot d$$

$$(ac) \cdot (b'd') = (a'c') \cdot (bd)$$

Thus
$(ac, bd) \sim (a'c', b'd')$.
Thus $(a,b)\cdot(c,d) \sim (a',b')\cdot(c',d')$.

10. It is $\frac{\bar{i}}{|i|^2} = \frac{-i}{1} = -i$.

11. $1-i = \sqrt{2}\cdot e^{\frac{7\pi}{4}i}$.

$r^3 e^{3i\theta} = \sqrt{2}\cdot e^{\frac{7\pi}{4}i}$

$r = 2^{1/6}$, $\theta = \frac{7\pi}{12}$, $z_1 = 2^{1/6} e^{i\frac{7\pi}{12}}$

$r^3 e^{3i\theta} = \sqrt{2}\cdot e^{\frac{15\pi}{4}i}$

$r = 2^{1/6}$, $\theta = \frac{15\pi}{12}$, $z_2 = 2^{1/6} e^{i\,15\pi/12}$

$r^3 e^{3i\theta} = \sqrt{2}\cdot e^{i\frac{23\pi}{4}}$

$r = 2^{1/6}$, $\theta = \frac{23\pi}{12}$ $z_3 = 2^{1/6} e^{i\,23\pi/12}$

12. The integers $\mathbb{Z}$ here complement the empty set. The empty set is declared to be open. So both $\mathbb{Z}$ and $\emptyset$ are open sets.
If U_α are open then each U_α has finite complement E_α. Then let $\mathcal{U} = \cup U_\alpha$.
We see that

$$^c\mathcal{U} = {}^c\left(\bigcup_\alpha U_\alpha\right) = \bigcap_\alpha {}^cU_\alpha = \bigcap_\alpha E_\alpha$$

which will be finite. So $\mathcal{U}$ is open.
Finite intersections work too. So it is a topology.

13. Suppose not. Then there are $e_j \in E$, $k_j \in K$ with $|e_j - k_j| \to 0$. Since K is compact, there is a convergent subsequence $\{k_{j_\ell}\}$ with $k_{j_\ell} \to k_0 \in K$. But then $\{k_{j_\ell}\}$ is bounded so $\{e_{j_\ell}\}$ is bounded. Hence e_{j_ℓ} will have a convergent subsequence $e_{j_{\ell_m}} \to e_0$.
And $|e_{j_{\ell_m}} - k_{j_{\ell_m}}| \to 0$. Thus $e_0 = k_0 \in E \cap K$.
Contradiction. So ε_0 exists.

14. $(1\underline{i} + 1\underline{j})(a\underline{1} + b\underline{i} + c\underline{j} + d\underline{k})$

$$= a\underline{i} - b\underline{1} + c\underline{k} - d\underline{j} + a\underline{j} - b\underline{k} - c\underline{1} + d\underline{i}$$

$$= (-b-c)\underline{1} + (a+d)\underline{i} + (-d+a)\underline{j} + (c-b)\underline{k}$$

So $-b-c = 1$, $a+d = 0$, $-d+a = 0$, $c-b = 0$
Hence $a = d = 0$, $b = -\frac{1}{2}$, $c = -\frac{1}{2}$.

The multiplicative inverse of $\underline{i} + \underline{j}$ is

$$-\frac{1}{2}\underline{i} - \frac{1}{2}\underline{j}$$