Math 310
December 14, 2020 Lecture

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Figure: This is your instructor.

More on the Cantor Set

Recall that, in the last lecture, we showed that the Cantor set has "zero length." So, in some sense, the Cantor set is small. Now we will show that, in another sense, the Cantor set is large.

Proposition: The Cantor set is uncountable.

Proof: We assign a binary "address" to each element of the Cantor set as follows. The address will be a sequence of 0s and 1s. Fix an element x of the Cantor set. Examine the second figure in the last lesson. The first entry of the address is a 0 if x is in the left component of I_1; otherwise the first entry is a 1. Now look at the third figure. An open segment is removed from the segment in the second figure in which x resides. Thus x is either in the resulting left component or in the resulting right component. As a result, the second address entry is either 0 or 1, respectively. Continue in this fashion, assigning the j^{th} address entry to be 0 or 1 according as x lies in either the left or right component of the divided interval in I_j.

We see that we may assign to each element of the Cantor set a sequence of 0s and 1s. Conversely, each sequence of 0s and 1s determines an element of the Cantor set using the same logic. We learned earlier in class that the set of all sequences of 0s and 1s is uncountable. Hence the Cantor set is uncountable. □

We conclude this discussion by constructing a counterintuitive function. Called the "Cantor–Lebesgue function," it has the property that it is continuous, its derivative is 0 at most points (i.e., on the complement of the Cantor set), yet the function is not constant. A graph of the function is exhibited in the next figure.

Proposition:

There exists a continuous function c with domain the interval $[0, 1]$ and range the interval $[0, 1]$ such that

(1) The function takes the values $c(0) = 0$ and $c(1) = 1$;

(2) The function c is monotone increasing;

(3) On the complement of the Cantor set, c is locally constant.

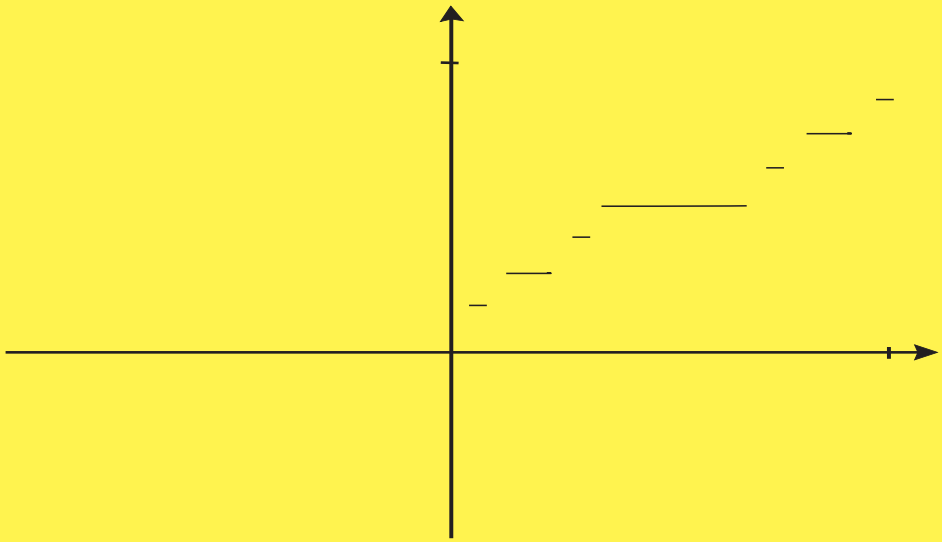


Figure: The Cantor–Lebesgue function.

Proof of the Proposition: Define the Cantor–Lebesgue function in this way:

$$c(x) = \begin{cases} \sum_{j=1}^{\infty} i_j \cdot (2^{-j})^{i_j} & \text{if} \quad x \in C \\ & \text{and } x \text{ has address } \{i_j\}_{j=1}^{\infty} \\ \sup\{c(t) : t \in C \text{ and } t \leq x\} & \text{if} \quad x \notin C\,. \end{cases}$$

Here $\sup$ is the abbreviation for "supremum," which is the same thing as "least upper bound." In the first line of the definition, on the right, we use the "address" (the sequence of 0s and 1s) that was our device for seeing that the Cantor set is uncountable.

Notice that we first define the Cantor–Lebesgue function for x in the Cantor set, then we define the function at points not in the Cantor set *in terms of* our earlier definition at points of the Cantor set.

This definition is rather technical. It can also be described in words. On the single interval that is removed in the construction of I_1, we declare the Cantor–Lebesgue function to have value $1/2$. On the pair of intervals removed in the construction of I_2, we declare the Cantor–Lebesgue function to have values $1/4$ (on the left) and $3/4$ (on the right). On the four intervals removed in the construction of I_3, we declare the Cantor–Lebesgue function to have values $1/8$, $3/8$, $5/8$, $7/8$. And so on. The values of the Cantor–Lebesgue function at the intermediate points of the Cantor set are obtained by interpolation.

It is clear from our rigorous definition of the Cantor–Lebesgue function that it is monotone increasing. For if $x_1 < x_2$ are elements of the Cantor set then the address of x_2 will have more 1's in it than the address of x_1, hence the sum defining the value of c will have more positive terms. Likewise, if $x_1 < x_2$ are *not* elements of the Cantor set, and have an element of the Cantor set between them, then the sup defining $c(x_2)$ will be a supremum over *more* numbers than the supremum defining $c(x_1)$, and hence will be larger. If there are *no* elements of the Cantor set between x_1 and x_2 then $c(x_1) = c(x_2)$. The verification of monotonicity for one point *in* the Cantor set and one point *not in* the Cantor set is immediate from the definitions.

The continuity of the Cantor–Lebesgue function follows from similar, but more technical, considerations. It is the proper subject matter of a course in real analysis. We refer the reader to [RUD] for the full details of this matter. □

Introduction to Topology

In mathematics and the physical sciences, it is important to be able to compare the *shapes* or *forms* of objects. Just what do we mean by "shape"? What does it mean to say that an object has a "hole" in it? Is the hole in the center of a basketball the same as the hole in the center of a donut? Is it correct to say that a ruler and a sheet of paper have the same shape—both are, after all, rectangles? What is a rigorous and mathematical means of establishing that two objects are equivalent from the point of view of shape or form?

A *topological space* is a set X together with a collection of subsets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ that we call the *open sets*. We assume that

(a) The entire space X is open. In other words, $X = U_{\alpha_0}$ for some α_0.

(b) The empty set $\emptyset$ is open. In other words, $\emptyset = U_{\alpha_1}$ for some α_1.

(c) If U_β are the open sets, then $\bigcup_\beta U_\beta$ is another open set ($\mathcal{U}$ is closed under the union operation).

(d) If U and V are open sets, then $U \cap V$ is an open set ($\mathcal{U}$ is closed under pairwise, indeed finite, intersection).

The entire subject of topology is based on the idea that if you know the open subsets of a space, then you know about its form. We sometimes write our topological space as $(X, \mathcal{U})$.

Example:

Let $X = \mathbb{R}$, the familiar real number system. Let us declare every open interval (a, b) to be an open set (here we allow a to be $-\infty$ or b to be $+\infty$). We also declare any union of such intervals to be an open set. It is easy to see that the collection $\mathcal{U}$ of all the open sets that we have described forms a topology (of course the empty set is the union of *no* open intervals). The entire real line is an open set according to our definition. Typical open sets are

$$U_1 = (-3, 5)$$

and

$$U_2 = (-1, 3) \cup (4, 9) \cup (16, 31)$$

and

$$U_3 = \cdots \cup (-3, -2) \cup (-1, 0) \cup (1, 2) \cup (3, 4) \cup \cdots$$

and

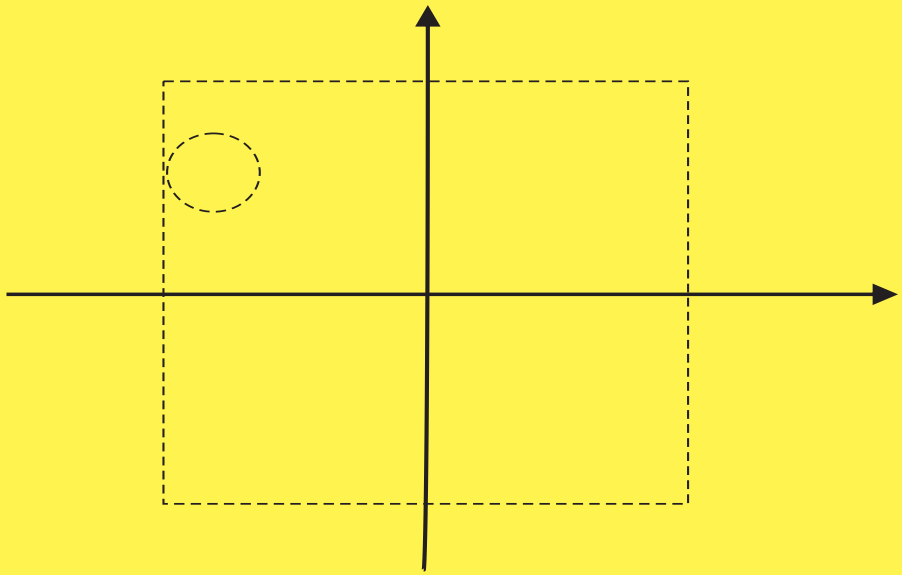
$$U_4 = (-\infty, -5) \cup (7, +\infty) .$$

We call this topology the *standard topology* or the *usual topology* or sometimes the *Euclidean topology* on the real line.

Example:

Let $X = \mathbb{R}^N$, the standard Euclidean space of N dimensions. We use coordinates to denote a point $x \in X$ by $x = (x_1, x_2, \ldots, x_N)$. Let us say that U is an open set if, whenever $x \in U$, then there is an $\epsilon > 0$ such that the ball $B(x, \epsilon) \equiv \left\{ t \in \mathbb{R}^N : |t - x| < \epsilon \right\}$ lies in U. We commonly refer to $B(x, \epsilon)$ as the *open ball* with center x and radius ϵ.

The entire space $U = \mathbb{R}^N$ is an open set according to this definition, and so is the empty set. But there are lots of other open sets as well. The set
$T = \{t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N : |t_j| < 1 \text{ for } j = 1, 2, \ldots, N\}$ is easily checked to be an open set. For, if $x = (x_1, \ldots x_N) \in T$, then let $\epsilon = \min\{|x_j \pm 1| : j = 1, \ldots, N\}$. Then $B(x, \epsilon) \subset T$ as required. See the figure.



The open sets are closed under the union operation and under finite intersection. So $\mathbb{R}^N$, equipped with the notion of openness specified here, is a topological space.

We call this topology the *standard topology* or the *usual topology* or sometimes the *Euclidean topology* on $\mathbb{R}^N$.

It is worth noting that the topology described here is "generated" by all the balls $B(x, \epsilon)$. Each such ball is open, just as the example set T above was open. Second, a union of such balls is clearly open. Third, if U is any open set according to our definition in the first paragraph, then each point in U has a ball about it (that lies in U). So U is a union of open balls. Thus the collection of unions of open balls is just the same as the collection of open sets.

It is also worth observing explicitly that the topology described here in case the dimension N is 1 is just the same as the topology considered in the previous example.

Remark:

For the record, we note that the *closed ball* with center x and radius ϵ in Euclidean space is given by

$$\overline{B}(x, \epsilon) = \{t \in \mathbb{R}^N : |t - x| \leq \epsilon\}.$$

See the figure.

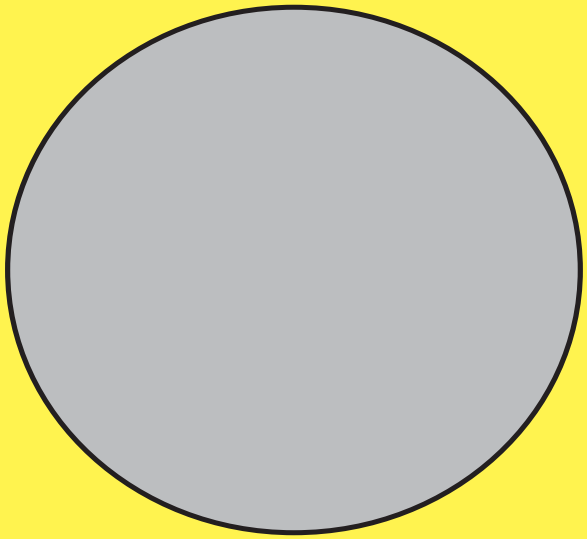


Figure: A closed ball.

Example:

Let X be the unit interval $[0, 1]$, and let the only open sets be the empty set $\emptyset$ and the entire interval $[0, 1]$. One may check directly that all the axioms for a topological space are satisfied. In particular, the union of the specified open sets can only be the empty set or the entire space. The same can be said for the intersection.