SOLUTIONS TO HW 9

MATH 309

Section 5.1

1. $\det\begin{bmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} = (3-\lambda)(8-\lambda) - 3\cdot 2$

$= 24 - 11\lambda + \lambda^2 - 6$

$= \lambda^2 - 11\lambda + 18$

$= (\lambda - 2)(\lambda - 9)$

$\lambda = 2, 9$ are eigenvalues. So yes.

4. $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Yes, it is an eigenvector with eigenvalue 2.

17. $\det\begin{bmatrix} 0-\lambda & 0 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{bmatrix} = -\lambda\left[(2-\lambda)(-1-\lambda) - 0\cdot 5\right]$

$= -\lambda\left[-2 - 2\lambda + \lambda + \lambda^2\right]$

$= +2\lambda + 2\lambda^2 - \lambda^2 - \lambda^3$

$= -\lambda^3 + \lambda^2 + 2\lambda$

$= -\lambda(\lambda^2 - \lambda - 2)$

$= -\lambda(\lambda - 2)(\lambda + 1)$

The eigenvalues are $0, 2, -1$.

Section 5.2

2. $\det\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 9 = 25 - 10\lambda + \lambda^2 - 9$

$= \lambda^2 - 10\lambda + 16$

$= (\lambda - 2)(\lambda - 8)$

Eigenvalues are $\lambda = 2, 8$.

10. $\det \begin{bmatrix} 0-\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} = -\lambda(\lambda^2-4) - 3(-3\lambda-2) + 1(6+\lambda)$

$= -\lambda^3 + 4\lambda + 9\lambda + 6 + 6 + \lambda$

$= -\lambda^3 + 14\lambda + 12$

15.

$$\det \begin{bmatrix} 4-\lambda & -7 & 0 & 2 \\ 0 & 3-\lambda & -4 & 6 \\ 0 & 0 & 3-\lambda & -8 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda)(3-\lambda)(1-\lambda)$$

Eigenvalues are $\lambda = 3$ with multiplicity 2
$\lambda = 1$ with multiplicity 1
$\lambda = 4$ with multiplicity 1

Section 5.3

7. $\det \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 6 \cdot 0 = 0$

So eigenvalues are $1, -1$.

$\lambda = 1$ $\quad \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 is free so $x_2 = \lambda$

$x_1 = \frac{1}{3} x_2 = \frac{1}{3}\lambda$

eigenvector is $\begin{bmatrix} \lambda/3 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda = -1$ $\quad \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 is free so $x_2 = \lambda$

$x_1 = 0$

eigenvector is $\begin{bmatrix} 0 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Section 5.4

2. $$\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

4. $T(\underline{p} + \underline{q}) = [\underline{p} + \underline{q}](0) - [\underline{p} + \underline{q}](1)t + [\underline{p} + \underline{q}](2)t^2$

$$= [p(0) - p(1)t + p(2)t^2] + [q(0) - q(1)t + q(2)t^2].$$

$$T(c\underline{p}) = c\underline{p}(0) - c\underline{p}(1)t + c\underline{p}(2)t^2$$

$$= c[p(0) - p(1)t + p(2)t^2]$$

Section 5.5

2. $\det \begin{bmatrix} -1-\lambda & -1 \\ 5 & -5-\lambda \end{bmatrix} = (-1-\lambda)(-5-\lambda) - 5(-1)$

$= 5 + 6\lambda + \lambda^2 + 5$

$= \lambda^2 + 6\lambda + 10 = 0$

$\lambda = \frac{-6 \pm \sqrt{36-40}}{2}$

$= \frac{-6 \pm 2i}{2} = -3 \pm i$

$\lambda = -3+i$ $\left[\begin{array}{cc|c} -1-(-3+i) & -1 & 0 \\ 5 & -5-(-3+i) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 is free so $x_2 = \lambda$

$(2-i)x_1 = x_2 = \lambda$

$x_1 = \frac{1}{2-i}\lambda = \frac{2+i}{5}\lambda$

eigenvector $= \begin{bmatrix} \frac{2+i}{5}\lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} \frac{2+i}{5} \\ 1 \end{bmatrix}$

$\lambda = -3-i$ $\left[\begin{array}{cc|c} -1-(-3-i) & -1 & 0 \\ 5 & -5-(-3-i) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 5 & -2+i & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 is free $x_2 = \lambda$

$(2+i)x_1 = x_2 = \lambda$

$x_1 = \frac{1}{2+i}\lambda = \frac{2-i}{5}\lambda$

eigenvector $= \begin{bmatrix} \frac{2-i}{5}\lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} \frac{2-i}{5} \\ 1 \end{bmatrix}$.

7. By Example 6,

$\lambda = \sqrt{3} \pm 1i$

$r = |\lambda| = \sqrt{3+1} = 2$

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} = 2 \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$\cos\varphi = \frac{\sqrt{3}}{2}$

$\sin\varphi = 1/2$

$\therefore \varphi = \pi/6.$

Section 6.1

2. $\underset{=}{w} \cdot \underset{=}{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 9+1+25 = 35$

$\underset{=}{x} \cdot \underset{=}{w} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 18 + 2 + (-15)$
$= 5$

$$\frac{\underset{=}{x} \cdot \underset{=}{w}}{\underset{=}{w} \cdot \underset{=}{w}} = \frac{5}{35} = \frac{1}{7}$$

6. $\underset{=}{x} \cdot \underset{=}{x} = 6 \cdot 6 + (-2)(-2) + 3 \cdot 3$
$= 36 + 4 + 9 = 49$

$$\left(\frac{\underset{=}{x} \cdot \underset{=}{w}}{\underset{=}{x} \cdot \underset{=}{x}} \right) \underset{=}{x} = \left(\frac{5}{49} \right) \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$$

10. $\frac{1}{\sqrt{9+36+9}} \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 3/54 \\ 6/54 \\ -3/54 \end{bmatrix} = \begin{bmatrix} 1/18 \\ 1/9 \\ -1/18 \end{bmatrix}$

15. $\begin{bmatrix} 8 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix} = -16 + 15 = -1 \neq 0$

not orthogonal

Section 6.2

2. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = -2 + 2 = 0$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = -5 + 4 + 1 = 0$

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 0 - 2 + 2 = 0$

This is an orthogonal set.

7. $\vec{u}_1 \cdot \vec{u}_2 = 0$. So this an orthogonal basis.

$$\left[\begin{array}{cc|c} 2 & 6 & 9 \\ -3 & 4 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 6 & 9 \\ 0 & 13 & 13/2 \end{array}\right]$$

$c_2 = \frac{1}{2}$

$c_1 = -3c_2 + \frac{9}{2}$

$= -\frac{3}{2} + \frac{9}{2} = 3$

So $\begin{bmatrix} 9 \\ -7 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

11. $\underline{p} = \dfrac{\begin{bmatrix}1\\7\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}{\begin{bmatrix}-4\\2\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}\cdot\begin{bmatrix}-4\\2\end{bmatrix}$

$= \frac{6}{20}\begin{bmatrix}-4\\2\end{bmatrix} = \frac{3}{10}\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}-6/5\\-3/5\end{bmatrix}$

Section 6.3

2. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 2 & 1 & 1 & 1 & 5 \\ 1 & -1 & -2 & 1 & -2 \\ 1 & 1 & -1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 5 & -1 & 3 & -3 \\ 0 & 1 & -3 & 2 & -6 \\ 0 & 3 & -2 & -1 & -2 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & -3 & 2 & -6 \\ 0 & 5 & -1 & 3 & -3 \\ 0 & 3 & -2 & -1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & -3 & 2 & -6 \\ 0 & 0 & 14 & -7 & 27 \\ 0 & 0 & 7 & -7 & 16 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & -3 & 2 & -6 \\ 0 & 0 & 7 & -7 & 16 \\ 0 & 0 & 0 & 7 & -5 \end{array}\right]$

$c_4 = -\frac{5}{7}$

$7c_3 = +7c_4 + 16 = -5 + 16 = 11 \Rightarrow c_3 = \frac{11}{7}$

$c_2 = 3c_3 - 2c_4 - 6 = \frac{33}{7} + \frac{10}{7} - 6 = \frac{1}{7}$

$c_1 = 2c_2 - c_3 + c_4 + 4 = \frac{2}{7} - \frac{11}{7} - \frac{5}{7} + \frac{28}{7} = \frac{14}{7} = 2$

So

$\underline{v} = 2\underline{u}_1 + \frac{1}{7}\underline{u}_2 + \frac{11}{7}\underline{u}_3 - \frac{5}{7}\underline{u}_4 = 2\underline{u}_1 + \left[\frac{1}{7}\underline{u}_2 + \frac{11}{7}\underline{u}_3 - \frac{5}{7}\underline{u}_4\right]$

4. $\underset{=}{u}_1 \cdot \underset{=}{u}_2 = 3\cdot(-4) + 4\cdot 3 + 0\cdot 0 = 0$

So this is an orthogonal set.

The projection of $\underset{=}{y}$ into $\text{Span}\{\underset{=}{u}_1, \underset{=}{u}_2\}$ is

$$\frac{\underset{=}{y}\cdot\underset{=}{u}_1}{\underset{=}{u}_1\cdot\underset{=}{u}_1}\underset{=}{u}_1 + \frac{\underset{=}{y}\cdot\underset{=}{u}_2}{\underset{=}{u}_2\cdot\underset{=}{u}_2}\underset{=}{u}_2$$

$$= \frac{24}{25}\begin{bmatrix}3\\4\\0\end{bmatrix} + \frac{-7}{25}\begin{bmatrix}-4\\3\\0\end{bmatrix} = \begin{bmatrix}4\\3\\0\end{bmatrix}$$

Section 6.4

4. $\underset{=}{v}_1 = \begin{bmatrix}3\\-4\\5\end{bmatrix}$

$$\underset{=}{v}_2 = \underset{=}{u}_2 - \frac{\underset{=}{u}_2\cdot\underset{=}{v}_1}{\underset{=}{v}_1\cdot\underset{=}{v}_1}\underset{=}{v}_1$$

$$= \begin{bmatrix}-3\\14\\-7\end{bmatrix} - \frac{-100}{50}\begin{bmatrix}3\\-4\\5\end{bmatrix} = \begin{bmatrix}-3\\14\\-7\end{bmatrix} + \begin{bmatrix}6\\-8\\10\end{bmatrix} = \begin{bmatrix}3\\6\\3\end{bmatrix}$$

Note that $\underset{=}{v}_1 \cdot \underset{=}{v}_2 = 0$.