MATH 309

Solutions to HW 4

Section 1.4

1. $\begin{bmatrix} -4 & 2 \\ 2 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$. This product is undefined since the matrix is 3×2 but the vector is a 3-vector (not a 2-vector).

2. This product is undefined for a similar reason

11. $\left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{array}\right]$ $x_3 = 1$, $x_2 = -3$, $x_1 = 0$. Sol. is $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \underline{x}$

4. $\begin{bmatrix} 8 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$

Section 2.5

1. $\left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 is free $x_3 = \lambda$

$x_2 = \frac{3}{4}x_3 = \frac{3}{4}\lambda$

$x_1 = \frac{5}{2}x_2 - 4x_3 = \frac{15}{8}\lambda - 4\lambda = \frac{-17}{8}\lambda$

So the solution set is

$$\left\{ \lambda \begin{bmatrix} -17/8 \\ 3/4 \\ 1 \end{bmatrix} \right\}.$$

3. $\left[\begin{array}{ccc|c} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -3 & 5 & -7 & 0 \\ 0 & -3 & 15 & 0 \end{array}\right]$

x_3 is free $\quad x_3 = \lambda$

$x_2 = 5x_3 = 5\lambda$

$x_1 = \frac{5}{3}x_2 - \frac{7}{3}x_3 = \frac{25}{3}\lambda - \frac{7}{3}\lambda = 6\lambda$

Solution set is $\left\{ \lambda \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$

5. $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -6 & 0 \\ -4 & -9 & 2 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 free $\quad x_3 = \lambda$

$x_2 = 2x_3 = -2\lambda$

$x_1 = -3x_2 - x_3 = 5\lambda$

Solution set is $\left\{ \lambda \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$

17. $\left[\begin{array}{cccccc|c} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$

x_6 is free $\quad x_6 = \lambda$

x_4 is free $\quad x_4 = \mu$

x_2 is free $\quad x_2 = \tau$

$x_5 = 4x_6 = 4\lambda$

$x_3 = x_6 = \lambda$

$x_1 = 4x_2 + 2x_3 - 3x_5 + 5x_6 = 4\tau + 2\lambda - 12\lambda + 5\lambda$

$= 4\tau - 5\lambda$

Solution set is

$$\left\{\begin{bmatrix} 4\tau - 5\lambda \\ \tau \\ \lambda \\ \mu \\ 4\lambda \\ \lambda \end{bmatrix}\right\} = \left\{\tau\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}\right\}$$

Section 1.7

1. $\left[\begin{array}{ccc|c} 5 & 7 & -2 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 5 & 7 & -2 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 is free $x_3 = 1$

$x_2 = x_3 = 1$

$x_1 = -2x_2 + x_3 = -1$

So $-1\begin{bmatrix}5\\1\\0\end{bmatrix} + 1\begin{bmatrix}7\\2\\-6\end{bmatrix} + 1\begin{bmatrix}-2\\-1\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

These are linearly dependent.

3. $\begin{bmatrix}1 & -3 & 0\\ -3 & 6 & 0\end{bmatrix} \to \begin{bmatrix}1 & -3 & 0\\ 0 & -3 & 0\end{bmatrix}$

$x_2 = 0$

$x_1 = 0$

So only the trivial solution.

These are linearly independent.

5. $\begin{bmatrix}0 & -8 & 5 & 0\\ 3 & -7 & 4 & 0\\ -1 & 5 & -4 & 0\\ 1 & -3 & 2 & 0\end{bmatrix} \to \begin{bmatrix}1 & -3 & 2 & 0\\ 0 & -8 & 5 & 0\\ -1 & 5 & -4 & 0\\ 3 & -7 & 4 & 0\end{bmatrix} \to \begin{bmatrix}1 & -3 & 2 & 0\\ 0 & -8 & 5 & 0\\ 0 & 2 & -2 & 0\\ 0 & 2 & -2 & 0\end{bmatrix}$

$\to \begin{bmatrix}1 & -3 & 2 & 0\\ 0 & -8 & 5 & 0\\ 0 & 2 & -2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & -3 & 2 & 0\\ 0 & 0 & -3 & 0\\ 0 & 2 & -2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\to \begin{bmatrix}1 & -3 & 2 & 0\\ 0 & 2 & -2 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ $\quad x_3 = 0$, $x_2 = 0$, $x_1 = 0$

So only the trivial solution. These vectors are linearly independent.

8 $\left[\begin{array}{cccc|c} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ -3 & 7 & -1 & 2 & 0 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -3 & 3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & -2 & 8 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$

$x_4 = 0$

x_3 is free $x_3 = \lambda$

$x_2 = 4x_3 - 3x_4 = 4\lambda$

$x_1 = 3x_2 - 3x_3 + 2x_4$

$= 12\lambda - 3\lambda + 0 = 9\lambda$

Thus $9\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \underline{\underline{0}}$

These vectors are linearly dependent.

Section 1.8

1, image of $\underline{\underline{u}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

image of $\underline{\underline{v}} = \begin{bmatrix} a \\ b \end{bmatrix}$ is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$.

3. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$

$x_3 = 2 \quad x_2 = -2x_3 + 5 = 1$

$x_1 = 2x_3 - 1 = 3 \Rightarrow \underline{\underline{x}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

$\underline{\underline{x}}$ is unique

5. $\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right]$

x_3 is free $\quad x_3 = \lambda$

$x_2 = -2x_3 + 1 = -2\lambda + 1$

$x_1 = 5x_2 + 7x_3 - 2 = 5(-2\lambda+1) + 7\lambda - 2$

$= -3\lambda + 3$

$$\underline{x} = \begin{bmatrix} -3\lambda+3 \\ -2\lambda+1 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

$\underline{x}$ is not unique.

9. $\left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{array}\right]$

$$\to \left[\begin{array}{cccc|c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_4 is free $\quad x_4 = \lambda$

x_3 is free $\quad x_3 = u$

$x_2 = 4x_3 - 3x_4 = 4u - 3\lambda$

$x_1 = 4x_2 - 7x_3 + 5x_4 = 4(4u - 3\lambda) - 7u + 5\lambda$

$= 9u - 7\lambda$

$$\underline{x} = \begin{bmatrix} 9u-7\lambda \\ 4u-3\lambda \\ u \\ \lambda \end{bmatrix} = u \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Section 7.9

1. The standard matrix is

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

Indeed, $A\underline{e}_1 = \begin{bmatrix} 2 & -5 \\ 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

$$A\underline{e}_2 = \begin{bmatrix} 2 & -5 \\ 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

2. $A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}$

Indeed, $A\underline{e}_1 = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$A\underline{e}_2 = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A\underline{e}_3 = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

7. $\underline{e}_1 \to \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $\underline{e}_2 \to \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

So $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

17. $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

$T(\underline{e}_1) = (0, 1, 0, 0)$

$T(\underline{e}_2) = (0, 1, 1, 0)$

$T(\underline{e}_3) = (0, 0, 1, 1)$

$T(\underline{e}_4) = (0, 0, 0, 1)$.

So. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$$