Math 233
Krantz

Spring, 2017
April 24, 2017

Practice Final Exam

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice or the TRUE/FALSE questions. For the questions that require a written answer, provide a *complete solution.* If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 200 points.

(15 points) **1.** Calculate the tangent plane to the graph of the function $f(x, y) = x^2 + 2xy^2$ at the point $(2, 1, 8)$.

$\underline{n} = \langle f_x, f_y, -1 \rangle = \langle 2x + 2y^2, 4xy, -1 \rangle.$

At $(2,1,8)$, $\underline{n} = \langle 6, 8, -1 \rangle.$

Tangent plane is

$\langle 6, 8, -1 \rangle \cdot \langle x-2, y-1, z-8 \rangle = 0$

$(6x - 12) + (8y - 8) + (-z + 8) = 0$

$6x + 8y - z = 12.$

(15 points) **2.** Locate and identify the local maxima and local minima and saddle points of the function $f(x, y) = x^2 + 3y^3 - 3y^2 - 4x + 2$.

$\nabla f = \langle 2x-4, 9y^2-6y \rangle = \langle 0, 0 \rangle$

$2x-4 = 0 \Rightarrow x = 2$

$9y^2-6y = 0 \Rightarrow 3y(3y-2) = 0 \Rightarrow y = 0, 2/3$

Critical points $(2,0), (2, 2/3)$.

$D = \det \begin{pmatrix} 2 & 0 \\ 0 & 18y-6 \end{pmatrix} = 36y - 12$

At $(2,0)$, $D = -12 < 0$, saddle point

At $(2, 2/3)$, $D = 12 > 0$, $f_{xx} > 0$, $f_{yy} > 0$ so local min.

(15 points) **3.** Find the extrema of the function $f(x, y) = x - y^2$ subject to the constraint $g(x, y) = x^2 + y^2 = 9$.

$\nabla f = \lambda \nabla g \Rightarrow \langle 1, -2y \rangle = \lambda \langle 2x, 2y \rangle$

$1 = 2\lambda x$

$-2y = 2\lambda y \Rightarrow 2y(-1-\lambda) = 0 \Rightarrow y = 0$ or $\lambda = -1$.

If $\lambda = -1$ then $x = -\frac{1}{2}$, $y = \pm \frac{\sqrt{35}}{2}$. Crit. pts. $\left(-\frac{1}{2}, \frac{\sqrt{35}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{35}}{2}\right)$

If $y = 0$ then $x = \pm 3$. Crits pts. $(3,0), (-3,0)$.

$f\left(-\frac{1}{2}, \frac{\sqrt{35}}{2}\right) = -\frac{37}{4}$, $f\left(-\frac{1}{2}, -\frac{\sqrt{35}}{2}\right) = -\frac{37}{4}$

min min

$f(3,0) = 3$, $f(-3,0) = -3$

max N/A

(15 points) **4.** The vector field $\mathbf{F}(x, y) = \langle \sin y + y^2, x \cos y + 2xy \rangle$ is known to be conservative. Find a potential function u.

$$\frac{\partial u}{\partial x} = \sin y + y^2 \Rightarrow u(x,y) = x \sin y + xy^2 + \varphi(y)$$

$$x\cos y + 2xy = \frac{\partial u}{\partial y} = x\cos y + 2xy + \varphi'(y) \Rightarrow \varphi'(y) = 0$$

$$\Rightarrow \varphi(y) = C.$$

$$\text{So } u(x,y) = x\sin y + xy^2 + C.$$

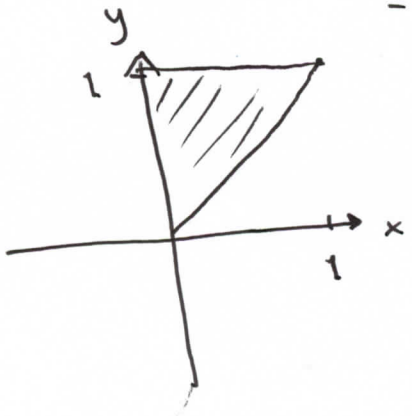
(20 points) **5.** Reverse the order of integration in order to evaluate the following double integral:

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx dy\,.$$

$$= \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 \frac{\sin x}{x} \cdot y \Big]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot (x - 0)\, dx = \int_0^1 \sin x\, dx = -\cos x \Big]_0^1$$

$$= -\cos 1 + \cos 0 = 1 - \cos 1.$$



(15 points) **6.** Calculate the volume of the solid that lies below the graph of $f(x,y) = 1+4y^2$ and over the region in the xy-plane bounded by $y = x^2 - 9$ and $y = -x^2 + 9$.

$$\int_{-3}^{3}\int_{x^2-9}^{9-x^2} 1+4y^2\,dy\,dx = \int_{-3}^{3} y + \frac{4}{3}y^3 \Big]_{y=x^2-9}^{y=9-x^2} dx$$

$$= \int_{-3}^{3} \left[(9-x^2) + \frac{4}{3}(9-x^2)^3\right] - \left[(x^2-9) + \frac{4}{3}(x^2-9)^3\right] dx$$

$$= \int_{-3}^{3} 9-x^2 + \frac{4}{3}(729 - 243x^2 + 27x^4 - x^6) - x^2 + 9 - \frac{4}{3}(x^6 - 27x^4 + 243x^2 - 729)\,d$$

$$= \int_{-3}^{3} 1962 - 650x^2 + 72x^4 - \frac{8}{3}x^6\,dx = 1962x - \frac{650}{3}x^3 + \frac{72}{5}x^5 - \frac{8}{21}x^7 \Big]_{-3}^{3}$$

$$= \left(5886 - 5850 + \frac{17496}{5} - \frac{17496}{21}\right) - \left(-5886 + 5850 - \frac{17496}{5} + \frac{17496}{21}\right)$$

$$= 283716/105.$$

(15 points) **7.** What are the parametric equations of the normal line to the surface $x^2 + y^2 + 4z^2 = 6$ at the point $(1,1,1)$?

$$\underline{n} = \langle 2x, 2y, 8z \rangle$$

At $(1,1,1)$, $\underline{n} = \langle 2, 2, 8 \rangle$.

Line is

$$x = 1 + 2t$$

$$y = 1 + 2t$$

$$z = 1 + 8t$$

(15 points) **8.** Calculate the curl of the vector field $\mathbf{F}(x, y, z) = x^2 z\mathbf{i} - xyz\mathbf{j} + yx^2\mathbf{k}$.

$$\operatorname{curl} \underset{\sim}{F} = \det \begin{pmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 z & -xyz & yx^2 \end{pmatrix} = \underset{\sim}{i}(x^2 + xy) - \underset{\sim}{j}(2xy - x^2) + \underset{\sim}{k}(-yz + 0)$$

$$= (x^2 + xy)\underset{\sim}{i} + (x^2 - 2xy)\underset{\sim}{j} - yz\underset{\sim}{k}$$

(15 points) **9.** Determine whether the vector field $\mathbf{F}(x, y) = (y^2 - x^2)\mathbf{i} + (2xy - x^2)\mathbf{j}$ is closed. If it is, then find a potential function.

$$\frac{\partial N}{\partial x} = 2y - 2x \qquad \frac{\partial M}{\partial y} = 2y$$

Those are unequal so not closed.

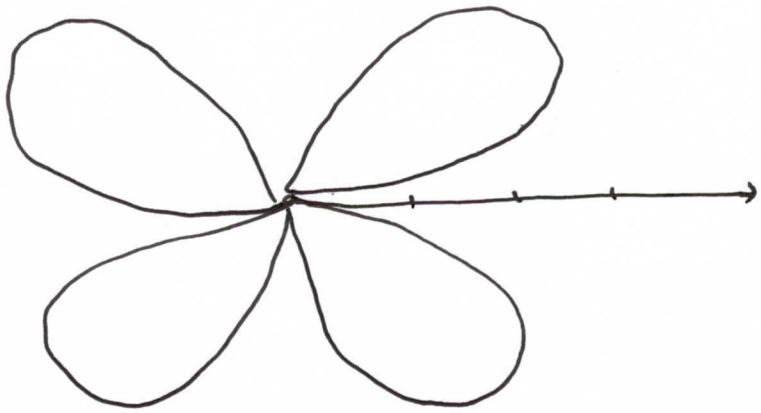
20 points) **10.** Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y) = xy^2\mathbf{i} + x\mathbf{j}$ and $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $1 \le t \le 2$.

$$\int_C \underline{F} \cdot d\underline{r} = \int_1^2 t \cdot t^4 \cdot 1 + t \cdot 2t \, dt = \int_1^2 t^5 + 2t^2 \, dt$$

$$= \frac{t^6}{6} + \frac{2}{3}t^3 \Big]_1^2 = \frac{64}{6} + \frac{16}{3} - \frac{1}{6} - \frac{2}{3}$$

$$= \frac{64 + 32 - 1 - 4}{6} = \frac{91}{6}.$$

20 points) **11.** Sketch the curve $r = 3\sin(2\theta)$ in polar coordinates.



20 points) **12.** Use Green's theorem to calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ by integrating over the region $\mathcal{R}$ that C bounds. Here $\mathbf{F}(x,y) = 4y\mathbf{i} + 4x\mathbf{j}$ and C is given by $\mathbf{r}(t) = 2\sin t\mathbf{i} + 3\cos t\mathbf{j}$, $0 \le t \le 2\pi$.

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

$$= \iint_R 4 - 4\, dA = 0.$$