SOLUTIONS TO PRACTICE MIDTERM 2

1. $y'' + 5y' + 4y = 0$

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$

$r = -4, -1.$

So $y = Ae^{-4x} + Be^{-x}.$

2. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r = 3, 3,$

So $y = Ae^{3x} + Bxe^{3x}.$

3. $y'' + 5y' + 6y = 0, \; y(0) = 0, \; y'(0) = 1.$

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -3, -2.$

$y = Ae^{-3x} + Be^{-2x}$

$0 = y(0) = A + B$

$1 = y'(0) = -3Ae^{-3\cdot 0} + B(-2)e^{-2\cdot 0}$

$0 = A + B \Rightarrow 0 = 2A + 2B$

$1 = -3A - 2B \qquad 1 = -3A - 2B$

$1 = -A$

So $A = -1$, $B = 1$

Sol. to i.v.p. is $y = -e^{-3x} + e^{-2x}$

4. $y'' + 4y' + 5 = 0$

$r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$

So $y = Ae^{-2x}\cos x + Be^{-2x}\sin x.$

5. $y'' - 7y' + 12y = 12x.$

Associated homogeneous is

$y'' - 7y' + 12y = 0$

$r^2 - 7r + 12 = 0$

$(r-3)(r-4) = 0$

$r = 3, 4$

$y = Ae^{3x} + Be^{4x}$

For a particular solution guess $y_0 = \alpha x^2 + \beta x + \gamma$

So $y_0' = 2\alpha x + \beta$

$y_0'' = 2\alpha$

$(2\alpha) - 7(2\alpha x + \beta) + 12(\alpha x^2 + \beta x + \gamma) = 12x$

$12\alpha x^2 + (-14\alpha + 12\beta)x + (2\alpha - 7\beta + 12\gamma) = 12x$

$$\left.\begin{array}{l} 12\alpha = 0 \\ 2\alpha - 7\beta + 12\gamma = 0 \\ -14\alpha + 12\beta = 12 \end{array}\right\} \begin{array}{l} \alpha = 0 \\ \beta = 1 \\ \gamma = 7/12 \end{array}$$

So a particular solution is $y_0 = x + \frac{7}{12}$

General solution is

$$y = \left(x + \frac{7}{12}\right) + Ae^{3x} + Be^{4x}.$$

6. $y'' - 3y' + 2y = e^{3x}$

Associated homogeneous is

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1, 2.$

Solutions are $y_1 = e^x, y_2 = e^{2x}.$

$$v_1' e^x + v_2' e^{2x} = 0 \quad (\star)$$

$$v_1' e^x + v_2' 2e^{2x} = e^{3x}$$

Subtract first equation from second to get

$$v_2' e^{2x} = e^{3x}$$

$$v_2' = e^x \quad (*)$$

$$v_2 = e^x$$

Plug $(*)$ into $(\star)$ to get

$$v_1' e^x + e^x e^{2x} = 0$$

$$v_1' e^x = -e^{3x}$$

$$v_1' = -e^{2x}$$

$$v_1 = -\frac{1}{2} e^{2x}.$$

So a particular solution is $y_0 = -\frac{1}{2} e^{2x} e^x + e^x e^{2x} = \frac{1}{2} e^{3x}.$

The general solution of the ODE is

$$y = \frac{1}{2} e^{3x} + Ae^x + Be^{2x}.$$

7. $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$

$y_1 = x$ is a solution.

For a second solution,

$$v = \int \frac{1}{x^2} e^{-\int \frac{1}{x}dx}\, dx = \int \frac{1}{x^2}\cdot\frac{1}{x}dx = \int x^{-3}dx = -\frac{1}{2}x^{-2},$$

So $y_2 = -\frac{1}{2}x^{-2}\cdot x = -\frac{1}{2}x^{-1}$ is a second solution.

General solution is $y = Ax + B\cdot(-\frac{1}{2}x^{-1}) = Ax + Cx^{-1}$.

8. $y' - 2xy = 0$.

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - 2x\sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} 2a_j x^{j+1} = 0$$

$$\sum_{j=-1}^{\infty} (j+2)a_{j+2}x^{j+1} - \sum_{j=0}^{\infty} 2a_j x^{j+1} = 0$$

$$1\cdot a_1 x^0 + \sum_{j=0}^{\infty}\left[(j+2)a_{j+2} - 2a_j\right]x^{j+1} = 0$$

Let $a_0 = C$

$a_1 = 0$

$(j+2)a_{j+2} - 2a_j = 0$ for $j \geq 0$

$$a_{j+2} = \frac{2}{j+2}a_j,\ j \geq 0$$

So $a_2 = \frac{2}{2}a_0 = C$

$a_3 = \frac{2}{3}a_1 = 0$

$$a_4 = \frac{2}{4} a_2 = \frac{1}{2} C$$

$$a_5 = \frac{2}{5} a_3 = 0$$

$$a_6 = \frac{2}{6} a_4 = \frac{1}{3 \cdot 2} C$$

$$a_7 = \frac{2}{7} a_5 = 0$$

$$a_8 = \frac{2}{8} a_6 = \frac{1}{4 \cdot 3 \cdot 2} C$$

etc.

So the solution is

$$C \sum_{j=0}^{\infty} \frac{1}{j!} x^{2j} = C e^{x^2}.$$

9. $y'' + xy' + y = 0$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}.$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + x \sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$2 \cdot 1 \cdot a_2 x^0 + a_0 x^0 + \sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} + j a_j + a_j \right] x^j = 0$$

$$2 a_2 + a_0 = 0$$

$$(j+2)(j+1) a_{j+2} + (j+1) a_j = 0, \quad j \geq 1$$

$$a_{j+2} = -\frac{a_j}{j+2}, \quad j \geq 1.$$

Let $a_0 = C$, $a_1 = D$

$$a_2 = -\frac{1}{2}a_0 = -\frac{C}{2}$$

$$a_3 = -\frac{a_1}{3} = -\frac{D}{3}$$

$$a_4 = -\frac{a_2}{4} = \frac{C}{4\cdot 2}$$

$$a_5 = -\frac{a_3}{5} = \frac{D}{5\cdot 3}$$

$$a_6 = -\frac{a_4}{6} = -\frac{C}{6\cdot 4\cdot 2}$$

$$a_7 = -\frac{a_5}{7} = -\frac{D}{7\cdot 5\cdot 3}$$

$$a_8 = -\frac{a_6}{8} = \frac{C}{8\cdot 6\cdot 4\cdot 2}$$

So the general solution is

$$y = C\left(1 - \frac{1}{2}x^2 + \frac{1}{4\cdot 2}x^4 - \frac{1}{6\cdot 4\cdot 2}x^6 + \frac{1}{8\cdot 6\cdot 4\cdot 2}x^8 - + \cdots\right)$$
$$+ D\left(x - \frac{1}{3}x^3 + \frac{1}{5\cdot 3}x^5 - \frac{1}{7\cdot 5\cdot 3}x^7 + - \cdots\right)$$

10. Ratio test:

$$\lim_{j\to\infty}\left|\frac{(-1)^{j+1}/(j+1)}{(-1)^j/j}\right| = \lim_{j\to\infty}\left|\frac{j}{j+1}\right| = 1.$$

So power series converges on $|x| < 1$.

At $x = 1$ have $\sum \frac{(-1)^j}{j}$ which converges by alternating series test.

At $x = -1$ have $\sum \frac{1}{j}$ which diverges.

So interval of convergence is $(-1, 1]$.

11. $(x-1)^2 y'' + (x-1) y' + y = 0$

$$y'' + \frac{1}{x-1} y' + \frac{1}{(x-1)^2} y = 0.$$

All points $x \neq 1$ are regular. Also

$(x-1) \cdot \frac{1}{x-1} = 1$ is nonsingular

and $(x-1)^2 \cdot \frac{1}{(x-1)^2}$ is nonsingular

So $x=1$ is a regular singular point.

12. $y'' + 4y = 0, \; y(0) = 1, \; y'(0) = 0.$

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

So $y = A \cos 2x + B \sin 2x$

$1 = y(0) = A \cos 2 \cdot 0 + B \sin 2 \cdot 0$

$= A$

$0 = y'(0) = -2A \sin 2 \cdot 0 + 2B \cos 2 \cdot 0$

$= 2B.$

So $A = 1, \; B = 0.$

Sol of i.v.p. is $y = \cos 2x.$