Math 217
Krantz

Spring, 2024
February 7, 2024

FIRST MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. You must show your work on each problem in order to receive full credit.

Be sure to ask questions if anything is unclear. There are twelve problems on the exam. Each problem on the exam is worth 10 points. So the exam in total is worth 120 points. You will have two hours to take this exam.

(10 points) **1.** Find the general solution of

$$y' = 2xy^2 .$$

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$-\frac{1}{y} = x^2 + C$$

$$\frac{1}{y} = -x^2 + D$$

$$y = \frac{1}{D - x^2}$$

(10 points) **2.** Solve the first-order linear equation

$$y' + 3x \cdot y = 2x\,.$$

$g(x) = e^{\int 3x\,dx} = e^{\frac{3}{2}x^2}$

$e^{\frac{3}{2}x^2} y' + 3x e^{\frac{3}{2}x^2} y = 2x e^{\frac{3}{2}x^2}$

$\left(e^{\frac{3}{2}x^2} y\right)' = 2x e^{\frac{3}{2}x^2}$

$e^{\frac{3}{2}x^2} y = \frac{2}{3} e^{\frac{3}{2}x^2} + C$

$y = \frac{2}{3} + C e^{-\frac{3}{2}x^2}$

(10 points) **3.** The solution of the initial value problem

$$\begin{aligned} y' + (2/x)y &= 3x^2 \\ y(1) &= 2 \end{aligned}$$

is what?

$g(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$x^2 y' + 2xy = 3x^4$

$(x^2 y)' = 3x^4$

$x^2 y = \frac{3}{5} x^5 + C$

$y = \frac{3}{5} x^3 + C x^{-2}$

(10 points) **4.** Solve the exact equation

$$(x^2 - 2y)dx + (y^2 - 2x)dy = 0\,.$$

$\frac{\partial f}{\partial x} = x^2 - 2y \Rightarrow f(x,y) = \frac{x^3}{3} - 2xy + \varphi(y)$

$y^2 - 2x = \frac{\partial f}{\partial y} = -2x + \varphi'(y) \Rightarrow \varphi'(y) = y^2 \Rightarrow \varphi(y) = \frac{y^3}{3} + C.$

So $f(x,y) = \frac{x^3}{3} - 2xy + \frac{y^3}{3} + C,$

$\therefore \frac{x^3}{3} - 2xy + \frac{y^3}{3} = D$

(10 points) **5.** Find the family of orthogonal trajectories to the curves $y = cx^3$.

$\frac{dy}{dx} = 3cx^2 = \frac{3y}{x}$

Family of orthogonal curves satisfies

$\frac{dy}{dx} = -\frac{x}{3y}$

$3y\,dy = -x\,dx$

$\frac{3y^2}{2} = -\frac{x^2}{2} + C$

$y^2 = -\frac{x^2}{3} + D$

$y = \pm\sqrt{-\frac{x^2}{3} + D}$

(10 points) **6.** Use the method of homogeneous equations to solve

$$(x-y)dx+(x+y)dy=0\,.$$

$$\frac{dy}{dx}=\frac{y-x}{x+y}=\frac{\frac{y}{x}-1}{1+\frac{y}{x}}$$

Let $z=\frac{y}{x}$, $y=zx$, $\frac{dy}{dx}=z+x\frac{dz}{dx}$

So $z+x\frac{dz}{dx}=\frac{z-1}{1+z}\Rightarrow x\frac{dz}{dx}=\frac{z-1}{z+1}-z=-\frac{z^2-1}{z+1}$

$\frac{(z+1)\,dz}{z^2+1}=-\frac{dx}{x}$ so $\frac{z}{z^2+1}dz+\frac{1}{z^2+1}dz=\frac{-dx}{x}\Rightarrow\frac{1}{2}\ln(z^2+1)+\tan^{-1}z=-\ln x+C$

Let $z=\frac{y}{x}$ to get $\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right)+\tan^{-1}\frac{y}{x}=-\ln x+C$.

(10 points) **7.** The equation

$$(xy+3x^3)dx+x^2dy=0$$

is *not* exact. Find an integrating factor for this equation.

$$g=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{x-2x}{x^2}=-\frac{1}{x}$$

$$u=e^{\int g\,dx}=e^{\int-\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}.$$

$$\frac{1}{x}(xy+3x^3)dx+\frac{1}{x}x^2dy=0$$

$$(y+3x^2)dx+x\,dy=0$$

is exact.

(10 points) **8.** Use the method of reduction of order to solve the differential equation

$$xy'' + y' = 2x.$$

$y' = p, y'' = p'$. So

$$xp' + p = 2x$$

$$p' + \frac{1}{x}p = 2$$

$$g = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$xp' + x \cdot \frac{1}{x}p = 2x \Rightarrow (xp)' = 2x$$

$$xp = x^2 + C$$

$$p = x + \frac{C}{x} \Rightarrow y = \frac{x^2}{2} + C \ln x$$

(10 points) **9.** Find the solution of the exact equation

$$4xydx + 2x^2dy = 0$$

that satisfies $y = 1$ when $x = 2$.

$$\frac{\partial f}{\partial x} = 4xy \Rightarrow f = 2x^2y + \varphi(y)$$

$$2x^2 = \frac{\partial f}{\partial y} = 2x^2 + \varphi'(y) \Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) = C.$$

So $f(x,y) = 2x^2y + C$.

Sol. is $2x^2y = D$

$2 \cdot 2^2 \cdot 1 = D \Rightarrow D = 8$.

So sol. of i.v.p. is $2x^2y = 8$.

(10 points) **10.** Use reduction of order to solve the differential equation

$$y'' = 4y'$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

$y' = p \Rightarrow y'' = p \cdot \frac{dp}{dy}$

$p \frac{dp}{dy} = 4p \Rightarrow \frac{dp}{dy} = 4 \Rightarrow p = 4y + C$

$y' = 4y + C$

$y' - 4y = C \Rightarrow (y'e^{-4x}) - 4e^{-4x}y = Ce^{-4x}$

$(ye^{-4x})' = Ce^{-4x}$

$ye^{-4x} = -\frac{C}{4}e^{-4x} + D$

$y = -\frac{C}{4} + De^{4x}$

(10 points) **11.** Solve the initial value problem

$$y' + 2xy = x \ , \ y(0) = 1 .$$

$y = e^{\int 2x dx} = e^{x^2}$

$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$

$(e^{x^2}y)' = xe^{x^2}$

$e^{x^2}y = \frac{1}{2}e^{x^2} + C$

$y = \frac{1}{2} + Ce^{-x^2}$

$1 = y(0) = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$\therefore$ sol. of i.v.p. is

$y = \frac{1}{2} + \frac{1}{2}e^{-x^2}$

(10 points) **12.** Solve the initial value problem

$$x^2y' + xy = 2x \qquad y(1) = 1\,.$$

$y' + \frac{1}{x}y = \frac{2}{x}$

$g = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$xy' + y = 2$

$(xy)' = 2$

$xy = 2x + C$

$y = 2 + \frac{C}{x}$

$1 = y(1) = 2 + \frac{C}{1} = 2 + C \Rightarrow C = -1$

So sol. of i.v.p. is

$y = 2 - \frac{1}{x}$