SOLUTIONS TO PRACTICE EXAM FOR FIRST MIDTERM

1. $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = \frac{dx}{x^2}$$

$$\ln|y| = -\frac{1}{x} + C$$

$$|y| = e^C \cdot e^{-1/x} = De^{-1/x}$$

For $y > 0$,

$$y = De^{-1/x}.$$

2. $y' - xy = 2x$

$$e^{\int a(x)dx} = e^{\int -x\,dx} = e^{-x^2/2}$$

$$e^{-x^2/2} y' - xe^{-x^2/2} y = 2xe^{-x^2/2}$$

$$(e^{-x^2/2} y)' = 2xe^{-x^2/2}$$

$$\int (e^{-x^2/2} y)'\,dx = 2\int xe^{-x^2/2}\,dx$$

$$e^{-x^2/2} y = -2e^{-x^2/2} + C$$

$$y = -2 + Ce^{x^2/2}$$

3. $y' - (1/x)y = x^3$, $y(2) = 3$

$e^{\int a(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|}$.

For $x > 0$, $\frac{1}{|x|} = \frac{1}{x}$.

$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}\cdot x^3$

$\left(\frac{1}{x}y\right)' = x^2$

$\int \left(\frac{1}{x}y\right)' dx = \int x^2 dx$

$\frac{1}{x}y = \frac{x^3}{3} + C$

$y = \frac{x^4}{3} + Cx$

$3 = y(2) = \frac{16}{3} + C\cdot 2$

$-\frac{7}{3} = 2C \Rightarrow C = -\frac{7}{6}$.

Sol. to IVP is $y = \frac{x^4}{3} - \frac{7}{6}$.

4. $(x^3+y)dx + (y^4+x)dy = 0$

$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. So exact

$\frac{\partial f}{\partial x} = M = x^3 + y \Rightarrow f(x,y) = \frac{x^4}{4} + xy + \varphi(y)$

$y^4 + x = N = \frac{\partial f}{\partial y} = x + \varphi'(y) \Rightarrow \varphi'(y) = y^4 \Rightarrow \varphi(y) = \frac{y^5}{5} + C$.

So $f(x,y) = \frac{x^4}{4} + xy + \frac{y^5}{5} + C$.

Sol. of ODE is $\frac{x^4}{4} + xy + \frac{y^5}{5} = D$.

5. $y = cx^3$

$$\frac{y}{x^3} = c$$

$$\frac{x^3 \frac{dy}{dx} - y \cdot 3x^2}{x^6} = 0$$

$$x^3 \frac{dy}{dx} - y \cdot 3x^2 = 0$$

$$\frac{dy}{dx} = \frac{3y}{x}$$

Orthogonal trajectory has slope

$$\frac{dy}{dx} = -\frac{x}{3y}$$

$$3y\,dy = -x\,dx$$

$$\frac{3}{2}y^2 = -\frac{x^2}{2} + C$$

$$y^2 = -\frac{x^2}{3} + \frac{2}{3}C = -\frac{x^2}{3} + D$$

$$y = \pm\sqrt{D - x^2/3}$$

6. $(x+y)\,dx + (y-x)\,dy = 0$

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$z + x\frac{dz}{dx} = \frac{1+z}{1-z}$$

$$x\frac{dz}{dx} = \frac{1+z}{1-z} - z = \frac{1+z-z+z^2}{1-z}$$

$$dz \cdot \frac{1-z}{1+z^2} = \frac{dx}{x}$$

$$\int \frac{dz}{1+z^2} - \int \frac{z\,dz}{1+z^2} = \int \frac{dx}{x}$$

$$\tan^{-1} z - \frac{1}{2}\ln(1+z^2) = \ln|x| + C$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C$$

For $x > 0$,

$$\tan^{-1}\frac{y}{x} - \ln\sqrt{1+\frac{y^2}{x^2}} = \ln x + C$$

$$\tan^{-1}\frac{y}{x} - \ln\sqrt{x^2+y^2} = C$$

7. $M = 3x^3y^2 - x\cos y$

$N = 2x^4y + x^2\sin y$

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{6x^3y + x\sin y - 8x^3y - 2x\sin y}{2x^4y + x^2\sin y}$$

$$= \frac{-2x^3y - x\sin y}{2x^4y + x^2\sin y} = -\frac{1}{x}.$$

$$\mu = e^{\int g\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|}.$$

For $x > 0$, $\mu = \frac{1}{x}$.

Multiply the ODE through by $\frac{1}{x}$:

$$(3x^2y^2 - \cos y)\,dx + (2x^3y + x\sin y)\,dy = 0.$$

This equation is now exact.

8. $y'' + xy' = 2x$

Let $p = y'$

$p' = y''$

$$p' + xp = 2x$$

$$e^{x^2/2}p' + xe^{x^2/2}p = 2xe^{x^2/2}$$

$$(e^{x^2/2}p)' = 2xe^{x^2/2}$$

$$e^{x^2/2}p = 2e^{x^2/2} + C$$

$$p = 2 + Ce^{-x^2/2}$$

$$y' = 2 + Ce^{-x^2/2}$$

$$y = \int 2 + Ce^{-x^2/2}\,dx + D.$$

$= c\int e^{-x^2/2}\,dx + 2x + D.$

We cannot evaluate $\int e^{-x^2/2}\,dx$

9. $4xy\,dx + 2x^2\,dy = 0$

$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$ so it is exact.

$\frac{\partial f}{\partial x} = M = 4xy \Rightarrow f(x,y) = 2x^2y + \varphi(y)$

$2x^2 = N = \frac{\partial f}{\partial y} = 2x^2 + \varphi'(y) \Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) = C_1.$

So $f(x,y) = 2x^2y + C.$

Sol. of ODE is

$2x^2y = D$

Now $y = 2$ when $x = 1$. So

$2 \cdot 1^2 \cdot 2 = D \Rightarrow D = 4.$

So sol of IVP is

$2x^2y = 4$

10. $y'' = -y'$. Let $p = y'$, $p' = y''$.

$p' = -p$

$\frac{dp}{dx} + p = 0$

$e^{\int a(x)\,dx} = e^{\int 1\,dx} = e^{x}$.

$e^{x}\frac{dp}{dx} + e^{x}p = 0$

$(e^{x}p)' = 0$

$e^{x}p = C$

$p = Ce^{-x}$

$\frac{dy}{dx} = Ce^{-x}$

$y = Ce^{-x} + D$, $y' = -Ce^{-x}$

$2 = y(1) = Ce^{-1} + D$

$3 = y'(1) = -Ce^{-1}$

$5 = D$

$2 = Ce^{-1} + 5$

$-3 = Ce^{-1} \Rightarrow C = -3e$

$\therefore$ sol. of IVP is $y = -3e \cdot e^{-x} + 5$.

11. $y' - xy = 3x$, $y(1) = 2$

$e^{\int a(x)\,dx} = e^{\int -x\,dx} = e^{-x^2/2}$.

$e^{-x^2/2}y' - xe^{-x^2/2}y = 3xe^{-x^2/2}$.

$$\left(e^{-x^2/2} y\right)' = 3xe^{-x^2/2}$$

$$e^{-x^2/2} y = -3e^{-x^2/2} + C$$

$$y = -3 + Ce^{x^2/2}.$$

$$2 = y(1) = -3 + Ce^{1/2}$$

$$5 = Ce^{1/2} \Rightarrow C = 5e^{-1/2}.$$

Sol. of IVP is $y = -3 + 5e^{-1/2}e^{x^2/2}$

22. $y' = x^3 y$

$$\frac{dy}{dx} = x^3 y$$

$$\frac{dy}{y} = x^3 dx$$

$$\ln|y| = \frac{x^4}{4} + C$$

$$|y| = e^C \cdot e^{x^4/4} = De^{x^4/4}.$$

$$y = Ee^{x^4/4}.$$