SOLUTIONS

Math 217
Krantz

Spring, 2024
May 3, 2024

FINAL EXAM

General Instructions: Read the statement of each problem carefully. You must show your work on each problem in order to receive full credit.

Be sure to ask questions if anything is unclear. Each problem on the exam is worth 10 points. So the exam in total is worth 120 points. You will have two hours to take this exam.

(10 points) **1.** The solution of the initial value problem

$$\begin{aligned} y' + (2/x)y &= x^2 \\ y(1) &= 1 \end{aligned}$$

is what?

Let $g(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$. Then

$x^2 y' + 2xy = x^4$

$(x^2 y)' = x^4$

$x^2 y = x^5/5 + C$

$y = x^3/5 + \frac{C}{x^2}$

$1 = y(1) = \frac{1}{5} + C \Rightarrow C = \frac{4}{5}$.

So solution of i.v.p is $y = \frac{x^3}{5} + \frac{4/5}{x^2}$.

(10 points) **2.** The equation

$$(xy - 3x^3)dx + x^2dy = 0$$

is *not* exact. Find an integrating factor for this equation.

Let $g(x) = \dfrac{\partial M/\partial y - \partial N/\partial x}{N} = \dfrac{x - 2x}{x^2} = \dfrac{-x}{x^2} = -\dfrac{1}{x}$.

Now $\mu(x) = e^{\int g(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$.

This is an integrating factor because

$$\frac{1}{x}(xy - 3x^3)dx + \frac{1}{x}x^2dy = 0$$

$$(y - 3x^2)dx + x\,dy = 0$$

is exact.

(10 points) **3.** Use the method of reduction of order to solve the differential equation

$$xy'' + y' = x\,.$$

$y' = p$, $y'' = p'$

So $xp' + p = x$

$p' + \frac{1}{x}p = 1$

$g(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x.$

$xp' + p = x$

$(xp)' = x$

$xp = \frac{x^2}{2} + C$

$p = \frac{x}{2} + \frac{C}{x}$

$y' = \frac{x}{2} + \frac{C}{x}$

$y = \frac{x^2}{4} + C\ln|x| + D$

(10 points) **4.** Find the solution of the following initial value problem:

$$y'' - 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 1.$$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 1, 2$

$y = Ae^x + Be^{2x}$

$1 = y(0) = A + B$

$1 = y'(0) = A + 2B$

$B = 0, \quad A = 1$

$\therefore y = e^x$ is the solution of the i.v.p.

(10 points) **5.** The equation $xy'' - y' = 0$ has the trivial solution $y_1(x) = x^2$. Find a second linearly independent solution y_2 and then find the general solution.

$$y'' - \frac{1}{x}y' = 0$$

$$v = \int \frac{1}{(x^2)^2} e^{-\int -\frac{1}{x}\,dx}\,dx$$

$$= \int \frac{1}{x^4} \cdot x\,dx = \int \frac{1}{x^3} = -\frac{1}{2}x^{-2}.$$

$$y_2 = vy_1 = -\frac{1}{2}x^{-2} \cdot x^2 = -\frac{1}{2}.$$

General solution is $y = Ax^2 + B\cdot\left(-\frac{1}{2}\right)$

$$= Ax^2 + C.$$

(10 points) **6.** Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' - 5y' + 6y = x\,.$$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r = 2, 3.$

So $y = Ae^{2x} + Be^{3x}$ is the solution of the homogeneous.

Guess $y_p = \alpha x^2 + \beta x + \gamma$

$y_p' = 2\alpha x + \beta$

$y_p'' = 2\alpha$

$2\alpha - 5(2\alpha x + \beta) + 6(\alpha x^2 + \beta x + \gamma) = x$

$6\alpha x^2 + (-10\alpha + 6\beta)x + (2\alpha - 5\beta + 6\gamma) = x$

$6\alpha = 0 \quad \Rightarrow \alpha = 0$

$-10\alpha + 6\beta = 1 \Rightarrow \beta = \frac{1}{6}$

$2\alpha - 5\beta + 6\gamma = 0 \quad \gamma = 5/36$

So $y_p(x) = \frac{1}{6}x + \frac{5}{36}$.

General solution is

$y = \left(\frac{1}{6}x + \frac{5}{36}\right) + Ae^{2x} + Be^{3x}.$

(10 points) **7.** Find a power series solution of the form $\sum_j a_j x^j$ for the following ODE:

$$y' = xy\,.$$

Identify the function it represents.

Guess $y = \sum_{j=0}^{\infty} a_j x^j$

$$y' = \sum_{j=1}^{\infty} j\, a_j x^{j-1}$$

$$\sum_{j=1}^{\infty} j\, a_j x^{j-1} - x \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=-1}^{\infty} (j+2)\, a_{j+2} x^{j+1} - \sum_{j=0}^{\infty} a_j x^{j+1} = 0$$

$$a_1 x^0 + \sum_{j=0}^{\infty} \left[(j+2)\, a_{j+2} - a_j \right] x^{j+1} = 0$$

$a_0 = C$

$a_1 = 0$

$(j+2)\, a_{j+2} - a_j = 0, \quad j \geq 0$

$a_{j+2} = \frac{a_j}{j+2}, \quad j \geq 0$

$a_2 = \frac{a_0}{2} = \frac{C}{2}$

$a_3 = \frac{a_1}{3} = 0$

$a_4 = \frac{a_2}{4} = \frac{C}{4 \cdot 2}$

$a_5 = \frac{a_3}{5} = 0$

$a_6 = \frac{a_4}{6} = \frac{C}{6 \cdot 4 \cdot 2}$

$$y = C \sum_{j=0}^{\infty} \frac{x^{2j}}{2^j j!} = \sum_{j=0}^{\infty} \frac{(x^2/2)^j}{j!} = e^{x^2/2}.$$

(10 points) 8. Calculate the Fourier series of the function $f(x) = 2x$, $-\pi \leq x \leq \pi$.

The function f is odd so the Fourier series will have only sine terms.

$$b_j = \frac{2}{\pi}\int_0^{\pi} 2x \sin jx \, dx$$

$$= \frac{2}{\pi}\left[2x\,\frac{-\cos jx}{j}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos jx}{j}\, 2\, dx\Bigg]$$

$$= \frac{2}{\pi}\left[2\pi\left(\frac{(-1)^j}{j}\right) - 0 + \frac{2\sin jx}{j^2}\right]_0^{\pi}\Bigg]$$

$$= \frac{4(-1)^j}{j}.$$

So Fourier series is $\displaystyle\sum_{j=1}^{\infty} \frac{4(-1)^j}{j} \sin jx.$

(10 points) **9.** Use the unit impulse function to solve the initial value problem

$$y'' + y' - 2y = 2t \ , \ y(0) = 0 \ , \ y'(0) = 0 \, .$$

$z(p) = p^2 + p - 2$

$h(t) = L^{-1}\left(\frac{1}{z(p)}\right) = L^{-1}\left(\frac{1}{p^2+p-2}\right) = L^{-1}\left(\frac{1}{(p+2)(p-1)}\right)$

$= L^{-1}\left(\frac{-1/3}{p+2} + \frac{1/3}{p-1}\right)$

$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}.$

Hence $y(t) = \int_0^t \left(-\frac{1}{3}e^{-2(t-\tau)} + \frac{1}{3}e^{(t-\tau)}\right) 2\tau \, d\tau$

$= -\frac{1}{3}e^{-2t}\int_0^t 2\tau e^{2\tau} d\tau + \frac{1}{3}e^{t}\int_0^t e^{-\tau} 2\tau \, d\tau$

$= -\frac{1}{3}e^{-2t}\left[2\tau \frac{e^{2\tau}}{2}\right]_0^t - \int_0^t 2\frac{e^{2\tau}}{2} d\tau \Big]$

$+\frac{1}{3}e^{t}\left[2\tau(-e^{-\tau})\right]_0^t + \int_0^t 2e^{-\tau} d\tau \Big]$

$= -\frac{1}{3}e^{-2t}\left[te^{2t} - 0 - \frac{e^{2\tau}}{2}\right]_0^t \Big]$

$+\frac{1}{3}e^{t}(-2te^{-t}) \quad + \frac{1}{3}e^{t}\left(-2e^{-\tau}\right)\Big]_0^t$

$= -\frac{1}{3}t + \frac{1}{3}e^{-2t}\left(\frac{e^{2\tau}}{2} - \frac{1}{2}\right) + \left(-\frac{2t}{3}\right) + \left(-\frac{2}{3} + \frac{2}{3}e^{t}\right)$

$= -\frac{1}{3}t + \frac{1}{6} - \frac{1}{6}e^{-2t} - \frac{2}{3} + \frac{2}{3}e^{t} - \frac{2t}{3}$

$= -\frac{1}{2} - t - \frac{1}{6}e^{-2t} + \frac{2}{3}e^{t}$

(10 points) **10.** Find the eigenvalues λ_n and eigenfunctions y_n for the equation $y'' + \lambda y = 0$ on the interval $[0, 2\pi]$ with the boundary conditions $y(0) = 0$, $y(2\pi) = 0$.

Only $\lambda > 0$ makes sense. Then the solutions of the ODE are

$$y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$$

But $y(0) = 0 \Rightarrow B = 0$.

So $y = A \sin \sqrt{\lambda} x$.

Now $y(2\pi) = 0 \Rightarrow 2\pi\sqrt{\lambda} = n\pi$

$$\Rightarrow \sqrt{\lambda} = \frac{n}{2}$$

$$\lambda = \frac{n^2}{4}, \quad n = 1, 2, \ldots$$

Then $y_n(x) = \sin \frac{n}{2} x$.

(10 points) **11.** Solve the Dirichlet problem on the unit disc with boundary function $f(\theta)$ equaling 1 on the top half of the boundary and equaling -1 on the bottom half of the boundary.

$$f(\theta) = \begin{cases} 1, & 0 \le \theta \le \pi \\ -1, & -\pi \le \theta < 0. \end{cases}$$

So f is odd. Thus.

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{n} + \frac{1}{n}\right]$$

If $n = 2k$ is even then $b_n = 0$

If $n = 2k-1$ is odd then $b_n = \frac{4}{n\pi}$.

So $f(\theta) = \frac{4}{\pi}\left(\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right)$

And the solution of the Dirichlet problem is

$$w(r,\theta) = \frac{4}{\pi}\left(r\sin\theta + \frac{r^3\sin 3\theta}{3} + \frac{r^5\sin\theta}{5} + \cdots\right).$$

(10 points) **12.** Consider the heat equation for $z(x,t)$ on $t \geq 0$ and $0 \leq x \leq \pi$:

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2},$$

with boundary conditions $z(0,t) = 0$ and $z(\pi,t) = 0$ for all t. Find the solution if the initial temperature is given by

$$z(x,0) = \sin x - \frac{1}{5}\sin 2x, \qquad 0 \leq x \leq \pi.$$

$z(x,t) = e^{-t^2}\sin x - \frac{1}{5}e^{-4t^2}\sin 2x$.