† includes work joint with
P. Griffiths, M. Green, G. Pearlstein, C. Doran

A tour of normal functions & algebraic cycles†

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Plan of talk: I'll describe some recent results on limits, singularities, and zero loci of normal functions, as well as connections (old & new) between normal functions and the Hodge Conjecture, Bloch-Beilinson filtrations, and (Homological) Mirror Symmetry.

I. Algebraic cycles and limits of Abel-Jacobi maps

The easiest theorem to start with is from

[Green/Griffiths/—, "Néron models and limits of AJ-mappings", Compositio (2009)].

Main idea: behavior of ① transcendental invariants
of ② cycles
on ③ varieties
as ④ the varieties undergo a degeneration.

The "classical cases" of these 4 basic concepts are limits/asymptotics
of ① abelian integrals
associated to ② divisors
on ③ algebraic curves
④ acquiring a nodal singularity,

which were already studied by Poincaré and used by Lefschetz to prove the only general known case of the Hodge conjecture.

Curves in the complex projective plane

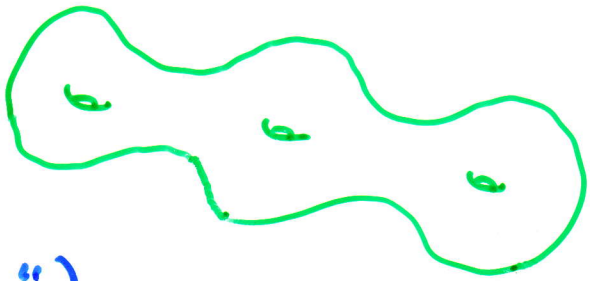
$$(\mathbb{C})\mathbb{P}^2 := \mathbb{C}^3 \setminus \{(0,0,0)\} \Big/ (x,y,z) \sim (\alpha x, \alpha y, \alpha z) \text{ for } \alpha \in \mathbb{C}^*$$

A degree d <u>algebraic curve</u> $C \subset \mathbb{P}^2$ is the zero-locus of a homogeneous polynomial $F(x,y,z)$ of degree d. <u>Singularities</u> of C are points where $F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$.

A nonsingular (smooth) curve is $\cong$ to a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$.

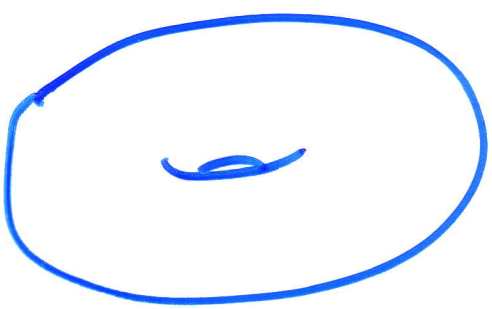
Ex/ smooth quartic curve

$0 = 5x^4 + 2y^3x + z^3y$

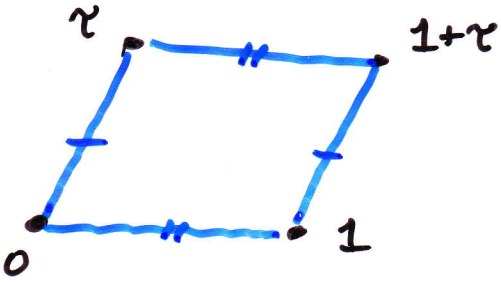


Ex/ smooth cubic ("elliptic") curve

$0 = t(x^3+y^3+z^3) - xyz$ $\quad (t \in \mathbb{C} \setminus \{0, 1, 1/3, e^{2\pi i/3}/3, e^{4\pi i/3}/3\})$



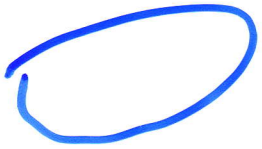
$\cong \mathbb{C}/\Lambda$ lattice

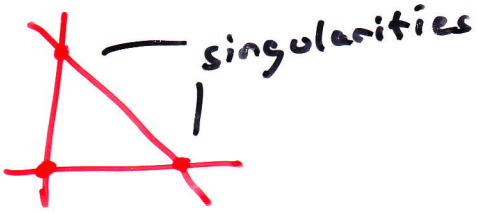


Degeneration of the cubic ($t \to 0$)

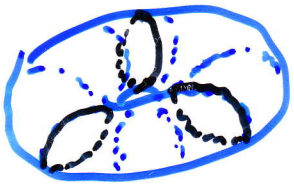
$t(x^3+y^3+z^3) - xyz = 0 \rightsquigarrow xyz = 0$

real solutions:

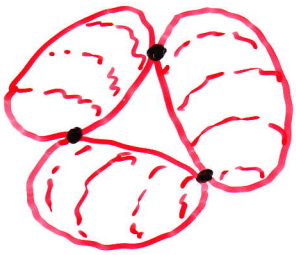




complex solutions:



pinch loops



3 $\mathbb{P}^1$'s (spheres)

Λ (lattice): $\mathbb{Z}\langle 1, \tau \rangle \xrightarrow{\tau \to i\infty} \mathbb{Z}$

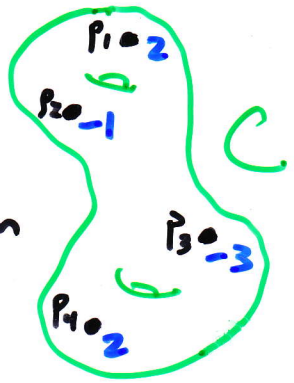
elliptic (Λ-periodic) functions:

$$\left.\begin{array}{l} \wp(u) + \frac{\pi^2}{3} \\ \frac{1}{u^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(u-\lambda)^2} - \frac{1}{\lambda^2} \right) \end{array}\right\} \rightsquigarrow \left\{\begin{array}{l} \pi^2 / \sin^2 \pi u \\ \| \\ \sum_{m \in \mathbb{Z}} \frac{1}{(u-m)^2} \end{array}\right.$$

$$C(\tau) \frac{\tilde{\sigma}(z - \tau/4)^2}{\tilde{\sigma}(z + \tau/4)^2} \rightsquigarrow \exp(4\pi i u)$$

and their inverses: abelian integrals $\rightsquigarrow$ log & inverse trig.

Divisors on a Riemann surface



A divisor on C is a finite formal sum of points $D = \sum n_i p_i$ with coefficients $n_i \in \mathbb{Z}$. $\quad \deg(D) := \sum n_i \in \mathbb{Z}$

Ex/ f = meromorphic function on $C \rightsquigarrow (f) := \sum_p \mathrm{ord}_p(f) \cdot p$

$\deg((f)) = 0$

When is a degree-0 divisor D of the form (f)?

Write $D = \sum (q_j - r_j)$, let $\{\omega_1, \dots, \omega_g\}$ be a basis for the holomorphic differentials on C, and consider the Abel-Jacobi vector

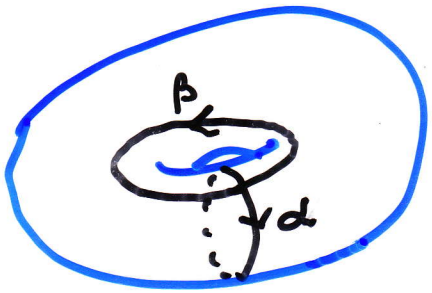
$$AJ(D) := \begin{pmatrix} \sum \int_{r_j}^{q_j} \omega_1 \\ \vdots \\ \sum \int_{r_j}^{q_j} \omega_g \end{pmatrix} \in \mathbb{C}^g .$$

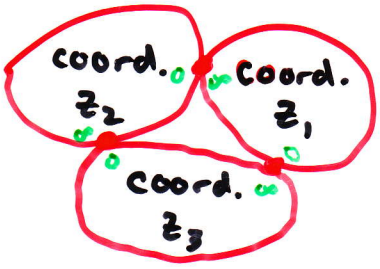
Abel's theorem: $D = (f) \iff AJ(D) = \begin{pmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_g \end{pmatrix}$ for a closed loop $\gamma \subset C$.

$$AJ_C : \frac{\mathrm{Div}^\circ(C)}{\mathrm{Rat}(C)} \xrightarrow{\cong} \frac{\Omega^1(C)^\vee}{\langle \int_{\text{loops}} \rangle} \cong \frac{\mathbb{C}^g}{\text{integral lattice}} =: J(C)$$

$\mathrm{Div}^\circ(C)/\mathrm{Rat}(C)$: divisors of fcns. $\quad J(C)$: Jacobian (complex torus)

Back to the degenerating elliptic curve

$E_{t\neq 0}$ 

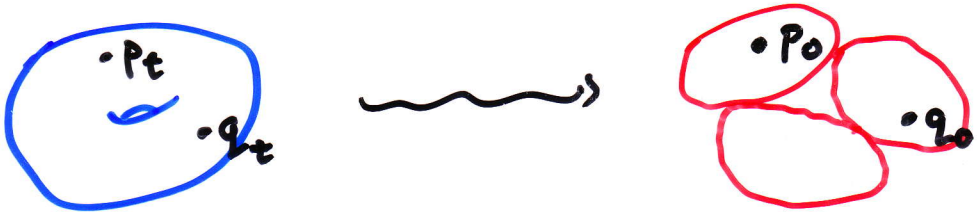
$\rightsquigarrow$ E_0 

$\omega_t \in \Omega^1(E_t) \rightsquigarrow \frac{1}{2\pi i}\frac{dz_j}{z_j}$ on each component

$\int_\alpha \omega_t = 1$, $\int_\beta \omega_t = \tau \longrightarrow i\infty$

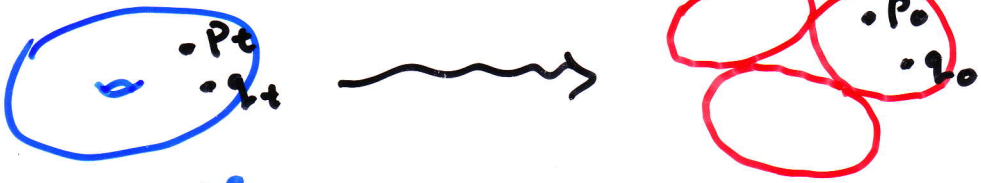
> Given $D_t = p_t - q_t$ varying "holomorphically" with E_t, how does $AJ_{E_t}(D_t)$ behave as $t \to 0$?

(A) It blows up



or,

(B) It has a limit



$$AJ_{E_t}(D_t) = \int_{q_t}^{p_t} \omega_t \in \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z} \rightsquigarrow AJ_{E_0}(D_0) = \int_{q_0}^{p_0} \frac{1}{2\pi i}\frac{dz_1}{z_1} = \frac{1}{2\pi i}\log\left(z_1(p_0)/z_1(q_0)\right) \in \mathbb{C}/\mathbb{Z}$$

Algebraic cycles

The common zero locus in $\mathbb{CP}^N$ of a collection of homogeneous polynomials is called a projective variety. A smooth variety X may be viewed as a compact complex manifold.

An algebraic cycle on X is a finite formal sum (w./integer coefficients) of subvarieties having some fixed codimension m in X. We write

$$\mathcal{Z} = \sum n_i V_i \in \mathcal{Z}^m(X).$$

A codimension m cycle $\mathcal{Z}$ has cohomology class

$$[\mathcal{Z}] \in H^{2m}(X, \mathbb{Z}).$$

Ex/ X algebraic curve ($\dim(X) = 1$)

$$\begin{array}{ccc} \mathcal{Z}^1(X) = \text{divisors} & & D \\ \downarrow & & \downarrow \\ H^2(X, \mathbb{Z}) & & [D] \\ \downarrow \cong & & \downarrow \\ \mathbb{Z} & & \deg(D). \end{array}$$

Example: a quintic 3-fold

$Y_t \subset \mathbb{P}^4$

$$\left\{ t(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5) - z_0 z_1 z_2 z_3 z_4 = 0 \right\}$$

Let $\{\zeta_i\}$ be 5th roots of unity. The subvarieties

$$L_0 = \left\{ \begin{matrix} z_0 = 0 \\ z_1 + \zeta_1 z_2 = 0 \\ z_3 + \zeta_2 z_4 = 0 \end{matrix} \right\}, \quad L_1 = \left\{ \begin{matrix} z_1 = 0 \\ z_0 + \zeta_3 z_2 = 0 \\ z_3 + \zeta_4 z_4 = 0 \end{matrix} \right\}$$

of Y_t are [projective] lines (there are 2875 such lines).

$L_0 - L_1 \in Z^2(Y_t)$ is a 1-cycle on Y_t.

We can degenerate Y_t by taking $t \to 0$:

$$Y_0 = \{ z_0 z_1 z_2 z_3 z_4 = 0 \} = \bigcup_{i=0}^{4} \underbrace{\{ z_i = 0 \}}_{\cong \mathbb{P}^3}$$

is singular, albeit in a rather nice way: it is a union of smooth varieties meeting transversely, i.e. a <u>normal crossing variety</u> (NCV).

L_0 and L_1 land in different components of the NCV Y_0.

Cycles $Z_0, Z_1 \in Z^m(X)$ are <u>rationally equivalent</u>, if they fit in an algebraic family of cycles Z_t parametrized by $t \in \mathbb{P}^1$ (= Riemann sphere).

[For divisors, this is just $Z_0 - Z_1 = (f)$.]

A basic, though somewhat mysterious, algebraic invariant of X is the <u>Chow group</u>

$$CH^m(X)_{hom} := \frac{Z^m(X)_{hom}}{\equiv_{rat}}$$

← cycles with $[Z] = 0$

One way to "detect" a cycle Z modulo $\equiv_{rat}$, is via Griffiths's <u>Abel-Jacobi map</u>

$$AJ_X : CH^m(X)_{hom} \xrightarrow{\text{(NOT } \simeq\text{)}} J^m(X)$$

← complex torus

which involves a substantial generalization of the "abelian integrals" above.

Now allow the cycle and variety to vary, with parameter $s \in S$, and assume X_s remains smooth. Then the <u>normal function</u>

$$\nu(s) := AJ_{X_s}(Z_s) \in J^m(X_s)$$

may be viewed as a section of a bundle $\mathcal{J}$ of complex tori over S.

Aside: Griffiths's AJ map

This is the basic "Hodge-theoretic" invariant attached to a cohomologically trivial algebraic cycle on a smooth projective variety $X/\mathbb{C}$.

For $\dim(X) = 2m-1$ and $Z \in CH^m(X)_{hom}$, let Γ be any piecewise C^∞ chain of real dimension $2m-1$ with

$$\partial \Gamma = Z.$$

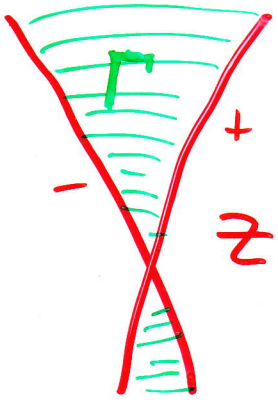
Then

$$AJ_X(Z) \in J^m(X) := \frac{H^{2m-1}(X,\mathbb{C})}{F^m H^{2m-1}(X,\mathbb{C}) + H^{2m-1}(X,\mathbb{Z})}$$

(complex torus)

is given by

$$\int_\Gamma (\cdot) \in \frac{\{F^m H^{2m-1}(X,\mathbb{C})\}^\vee}{\text{periods}}$$

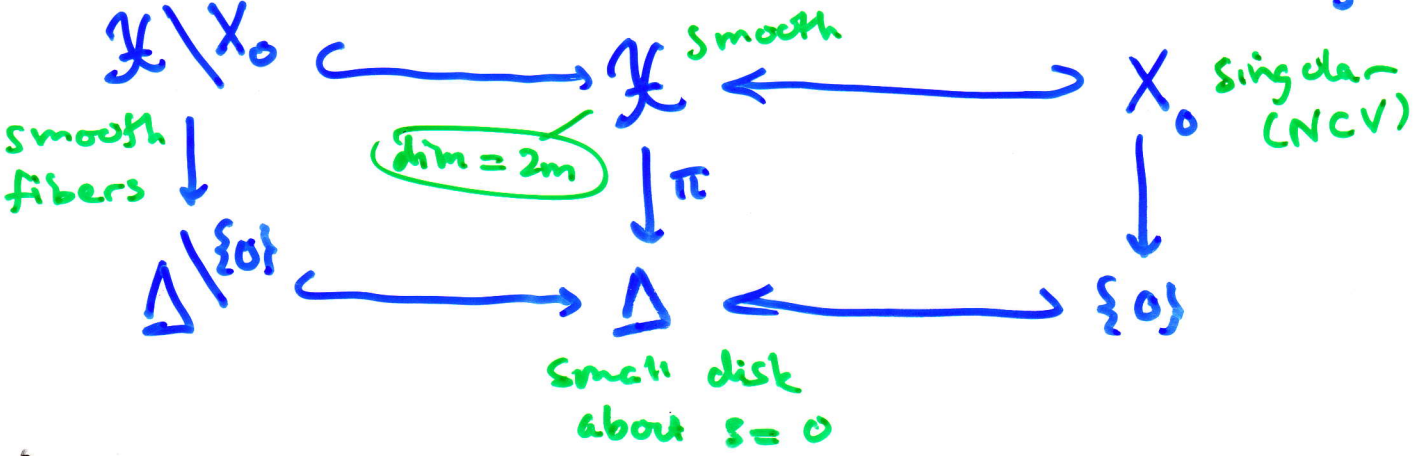


where $F^m H^{2m-1}(X,\mathbb{C})$ is the part of (de Rham) cohomology representable by forms with "at least m dz's" in local holomorphic coordinates.

Since a holomorphic map from $\mathbb{P}^1$ to a complex torus is constant, AJ is well-defined modulo $\equiv_{rat}$.

Limits of Abel-Jacobi maps

What if X_s becomes singular? Consider a "semistable degeneration of $(2m-1)$-folds" $X_s := \pi^{-1}(s)$



An algebraic cycle $\mathcal{Z} \in \mathcal{Z}^m(\mathcal{X})_{\text{hom}}$ meeting fibers nicely gives rise to a family

$$Z_s := \mathcal{Z} \cdot X_s \in \mathcal{Z}^m(X_s)_{\text{hom}}, \quad t \in \Delta;$$

this yields a normal function $\nu(s) \in \Gamma(\Delta \setminus \{0\}, \mathcal{J})$.

Zucker & El Zein (1984) constructed a bundle of complex Lie groups $\bar{\mathcal{J}} \to \Delta$ extending $\mathcal{J}$, and showed (using sheaf cohomology) that ν extends to a section $\bar{\nu} \in \Gamma(\Delta, \bar{\mathcal{J}})$. The important question (which we have answered) of interpreting

$$\lim_{s \to 0} AJ_{X_s}(Z_s) := \bar{\nu}(0) \in (\bar{\mathcal{J}})_0$$

geometrically in the non-classical case $m > 1$, was asked but left open in their work.

The singular variety $X_0 = \cup W_i \xhookrightarrow{\iota_0} \mathcal{X}$

has substrata (of codimension $\ell = 0, \ldots, 2m-1$)

$$W^{[\ell]} := \coprod_{|I| = \ell+1} W_I \quad \text{where} \quad W_I := \bigcap_{i \in I} W_i .$$

Using these one may explicitly write down Abel-Jacobi maps on the motivic cohomology of X_0:

$$H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{\text{hom}} \xrightarrow{AJ_{X_0}} J^m(X_0)$$

complex Lie group

(built out of algebraic K-groups of substrata $K_\ell(W^{[\ell]})$)

(built out of "regulators" = Chern-class maps on $K_\ell(W^{[\ell]})$)

The remaining two ingredients in the statement of our theorem below are:

$$CH^m(\mathcal{X})_{\text{hom}} \xrightarrow{\iota_0^*} H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{\text{hom}}$$

$$\mathcal{Z} \longmapsto Z_0$$

and a map

$$J(X_0) \xrightarrow{\Psi} (\bar{\mathcal{J}})_0 .$$

Theorem (GGK, 2007): $\lim_{s \to 0} AJ_{X_s}(Z_s) = \Psi(AJ_{X_0}(Z_0))$.

Corollary: $\lim_{s \to 0} AJ_{X_s}(Z_s) \in \operatorname{im}(\Psi) \subset (\bar{\mathcal{J}})_0$, and replacing $(\bar{\mathcal{J}})_0$ by $\operatorname{im}(\Psi)$ makes $\bar{\mathcal{J}}$ into a Hausdorff space. (The original version was non-separated.)

Back to the quintic 3-fold

The degeneration $Y_t \rightsquigarrow Y_0$ $(t \in \Delta)$ is not semistable, and one must replace Y_0 by a "larger" $\tilde{Y}_0 = \cup$ 5 $\mathbb{P}^3$'s blown up along Fermat curves.

2 ways to produce nontrivial family $Z_t \in CH^2(Y_t)$: $0 \neq_{rat}$

(A) Show $\nu(t)$ blows up at $t=0$

Ex/ Z_t = difference of lines landing in different components of $\tilde{Y}_0$.

Such "singularities of normal functions" are measured in this case by the finite group $(\mathbb{Z}/5\mathbb{Z})^3$.

(B) Show $\nu(t)$ has nontrivial limit using Theorem

$$\begin{array}{ccccc} H^4_{\mathcal{M}}(\tilde{Y}_0, \mathbb{Q}(2))_{hom} & \xrightarrow{AJ_{\tilde{Y}_0}} & J^2(\tilde{Y}_0) & \xrightarrow{\Psi} & (\overline{\mathcal{G}})_0 \\ \cup & & \cup & & \\ K_3^{ind}(\mathbb{C}) & \xrightarrow{\text{regulator}} & \mathbb{C}/(2\pi i)^2\mathbb{Z} & & \\ & & \downarrow \text{Re} & & \\ & & \mathbb{R} & & \end{array}$$

Ex/ $\exists$ family of 1-cycles Z_t with Z_0 "lying in the K_3^{ind} subgroup", and

$$\text{Re}(AJ_{X_0}(Z_0)) = D_2(\sqrt{-3}) \neq 0.$$

Intermezzo

Normal functions were introduced by Poincaré to study algebraic 1-cycles on complex algebraic surfaces. His ideas were used by Lefschetz — before "$H^1(\mathcal{O}) \cong H^{1,0}$" — to prove that the rational $(1,1)$ forms on an algebraic surface X represent the fundamental classes of algebraic cycles, along essentially the following (revisionist) lines:

$X \subset \mathbb{P}^N$ smooth projective surface

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi^{sm}} & U = \mathbb{P}^1 \setminus \Delta \\ \cap & & \cap \\ \mathcal{X} & \xrightarrow{\pi} & \mathbb{P}^1 \\ \downarrow \beta & & \\ X & & \end{array}$$

Lefschetz pencil of X

with
- $H^1 := R^1 \pi^{sm}_* \mathbb{Z}$ (local system)
- $\mathcal{F}_e := \pi_* \omega_{\mathcal{X}/\mathbb{P}^1}$ (Hodge bundle)

$\omega_{\mathcal{X}/\mathbb{P}^1}$: relative canonical bundle

An admissible normal function (ANF) is a section (over $\mathbb{P}^1$) of the extended Jacobian bundle defined by

$$0 \longrightarrow j_* H^1 \longrightarrow \mathcal{F}_e^\vee \longrightarrow \hat{\mathcal{J}} \longrightarrow 0 .$$

This induces (via the long exact sequence in sheaf cohomology)

$$0 \to \frac{H^0(F_e^\vee)}{H^0(j_* H)} \to H^0(\hat{Q}) \to \ker\{H^1(\mathbb{P}^1, j_* H) \to H^1(F_e^\vee)\} \to 0$$

$$\| \qquad\qquad !! \qquad\qquad \|$$

$$J^1(X) \hookrightarrow ANF \xrightarrow{[\cdot]} \frac{Hg^1(X)_{prim}}{\mathbb{Z}\langle [X_{s_0}] \rangle} \qquad (*)$$

constant NF's

$$\uparrow \beta^*$$

$$Hg^1(X)_{prim}$$

The following makes "Jacobi inversion with parameter" precise:

Poincaré Existence Theorem: For every ANF v,

$$\exists\ \zeta \in Z^1(X)_{prim} \text{ such that } v(s) = AJ(\zeta \cdot X_s) \quad \forall s \in \mathbb{P}^1$$

i.e. $\boxed{v = v_\zeta}$

Proof of Lefschetz (1,1): start with $\xi \in Hg^1(X)_{prim}$.

$$\beta^* \xi \underset{(*)}{=} [v] \underset{\text{P.E.T.}}{=\!=} [v_\zeta] = [\zeta] \quad ; \text{ applying } \beta_*,$$

$$\xi = \beta_* \beta^* \xi = [\beta_* \zeta]$$

$\beta_* \zeta$: alg. cycle on X.

□

In general, of course, one is after the

Hodge Conjecture: For any smooth projective variety X, $Hg^m(X) := F^m H^{2m}(X, \mathbb{Q}) \cap H^{2m}(X, \mathbb{Q})$ consists of fundamental classes of algebraic cycles.

An extension of the above argument works whenever $J^m(X_s) = J^m(X_s)_{alg}$ for a general member of the pencil, since then "Jacobi inversion with parameter" holds.

Examples: $m = 1$, X arbitrary
$m = 2$, X = cubic 4-fold (Zucker)

In general, this fails for $m > 1$, so that there is no hope in general to prove the HC by this method. What is left? Are normal functions worth considering? We have just seen that

- limits of NF's of geometric origin have interesting arithmetic content. This would not exist if Jacobi inversion worked!!

- In real life (linear systems, simplified polynomial moduli space, etc.) one has to deal with multiple parameters, and then nontension obstructions to limits — singularities of NF's — arise in codim. ≥ 2.

  These are connected with a different approach to the HC (Griffiths/Green) and GHC (—/Pearlstein).

- zero loci of NF's give algebraic subvarieties of the parameter space S (Brosnan/Pearlstein, Kato/Nakayama/Usui, M. Saito, C. Schnell).

  The key for proving this is to construct a Néron model graphing NF's (these constructions all generalize the one from Green/Griffiths/—) and show it is Hausdorff.

  * if $\dim\left(\begin{smallmatrix}\text{zero}\\ \text{locus}\end{smallmatrix}\right) > 0$ (hard to prove!) then the zero locus "produces" the singularities required to prove HC.
  * the field of definition of the ZL is connected to Bloch-Beilinson conjectures (—/Pearlstein, F. Charles).

- the homological mirror of a NF, yet to be completely understood, is at the centre of recent insights into open and local Gromov-Witten generating functions, and has the potential to bring the limiting arithmetic content of NF's to bear on symplectic geometry.

The open MS case (Morrison / Walcher)

The solution to the inhomogeneous Picard-Fuchs equation

$$(2\pi\sqrt{-1})^2 \left\{ \delta_z^4 - 5z \prod_{k=1}^{4} (5\delta_z + k) \right\} (\cdot) = -\frac{15}{4}\sqrt{z}$$ is:

(A model) the generating function of open GW invariants of the Fermat quintic 3-fold X° (after composing with the mirror map);

(B model) the membrane integral $\overbrace{\int_{\Gamma_s} \omega^{3,0}(s)}^{(*)}$ for a family of 1-cycles on [a double cover of] the mirror quintic family of CY 3-folds X_z.

(A ↔ B) The open GW #'s count holomorphic disks bounding on a Lagrangian (= real quintic) which is the homological mirror of the family of 1-cycles.

Remark: $(*)$ is a truncated normal function. The "full NF" would be $\begin{pmatrix} \int_{\Gamma_s} \omega^{3,0}(s) \\ \int_{\Gamma_s} \omega^{2,1}(s) \end{pmatrix}$. Limits at $z = 0$ & ∞ are essentially trivial (but this may not be true for other examples, one hopes). (& Q)

or quantum-cohomology

A physical interpretation of the full NF on the A-side, and what this has to do with the real quintic (i.e. an A-model MS map) is still lacking.

II. Néron models and zeroes of normal functions

Associated to a semistable degeneration $\mathcal{X} \supset X$, $\downarrow \pi$, $\Delta \supset \Delta^*$ there is a variation of Hodge structure (VHS) $/\Delta^*$ of weight $(2m-1)$: $\mathcal{H} = ((H, \mathcal{H}_{(0)}, F^\bullet), \nabla)$

where
- $H = R^{2m-1}\pi_* \mathbb{Z}$ — local system
- $\mathcal{H} = H \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$ — holomorphic vector bundle
- $\cup$
- $F^\bullet$ — decreasing filtration by holo. subbundles
- $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\Delta^*}$ — flat connection

such that (A) $(H_s, \mathcal{H}_s, F^\bullet_s)$ define a weight $(2m-1)$ HS $\forall s \in \Delta^*$:

$$\mathcal{H}_s = H_s \otimes \mathbb{C} \supset F^\bullet_s$$

$$F^p_s \oplus \overline{F^{2m-p}_s} = \mathcal{H}_s \quad (\forall p)$$

and (B) $\nabla(H) = 0$

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_{\Delta^*}$$ ← Griffiths transversality

Define the Jacobian bundle by $\boxed{0 \to H \to \frac{\mathcal{H}}{F^m} \to \mathcal{J} \to 0.}$

We have $\mathcal{J}_s = J^m(X_s) = \dfrac{H^{2m-1}(X_s, \mathbb{C})}{F^m H^{2m-1}(X_s, \mathbb{C}) + H^{2m-1}(X_s, \mathbb{Z})}$.

Any holomorphic vector bundle over Δ^* is trivial; any trivialization induces an extension to Δ. To do this for $\mathcal{H}$, let

$$T : \mathbb{H} \to \mathbb{H}$$

be the counterclockwise monodromy transformation, which is unipotent since the degeneration is semistable. Define

$$N := \log(T) = \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{k} (T-I)^k ,$$

the (nilpotent) monodromy logarithm. "Untwisting" the local system $\mathbb{H}_{\mathbb{Q}}$

$$\mathbb{H}_{\mathbb{Q}} \longmapsto \widetilde{\mathbb{H}}_{\mathbb{Q}} := \exp\left(-\frac{\log(s)}{2\pi i} N\right) \mathbb{H}_{\mathbb{Q}}$$

gives a basis for the canonical extension

$$\mathcal{H}_e := \widetilde{\mathbb{H}}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_\Delta .$$

N acts consistently on $\mathbb{H}_{\mathbb{Q}}$, $\widetilde{\mathbb{H}}_{\mathbb{Q}}$, $\mathcal{H}_e$, inducing a monodromy weight filtration $W_\bullet$ which is determined uniquely by the properties

(i) $\{0\} \subset W_0 \subset \cdots \subset W_{4m-2} = \text{everything}$

(ii) $N(W_k) \subset W_{k-2} \quad (\forall\, k)$

(iii) $N^k : Gr^W_{2m-1+k} \xrightarrow{\cong} Gr^W_{2m-1-k} \quad (\forall\, k)$.

Schmid: $F^\bullet$ extends to $F_e^\bullet \subset \mathcal{H}_e$ and the data $((\tilde{H}_{\mathbb{Q}})_0, (\mathcal{H}_e)_0, (F_e^\bullet)_0, W_\bullet) =:$ define a (limit) mixed Hodge structure (LMHS) $H^{2m-1}_{lim}(X_s)$ — taking $Gr^W_i H_{lim}$ yields a HS of weight i.

Now we get the extended Jacobian bundles

$$0 \to j_* H_{\mathbb{Z}} \to \mathcal{H}_e / F_e^m \to \bar{\mathcal{Q}} \to 0$$

$(j : \Delta^* \hookrightarrow \Delta)$

of Zucker.

$\hat{\mathcal{Q}} \subset \bar{\mathcal{Q}}$ is obtained by replacing

$$(\bar{\mathcal{Q}})_0 = \text{“}J(H^{2m-1}_{lim})\text{”}$$

by

$$J(\ker(N)) := \frac{\ker(T-I) \subset H^{2m-1}_{lim}(X_s, \mathbb{C})}{F^m_{lim} \ker(T-I) + \ker(T-I)_{\mathbb{Z}}}.$$

Examples:

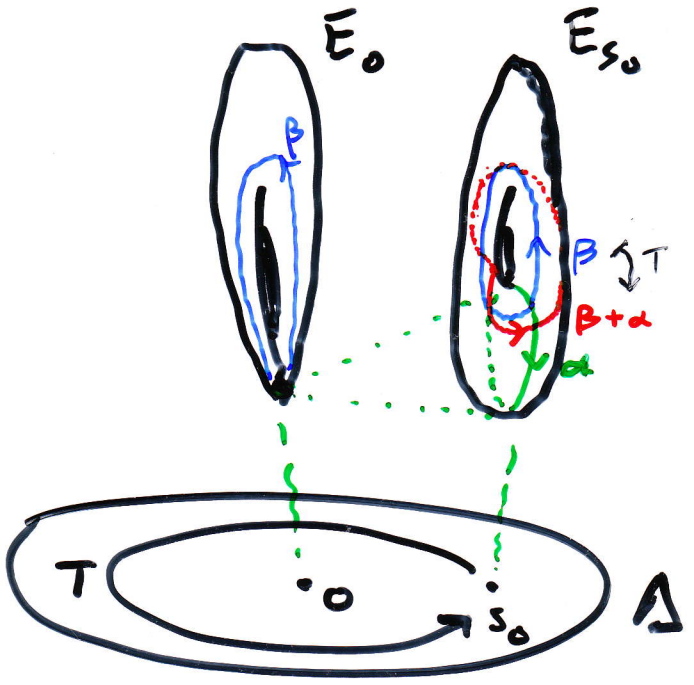
(I) $\mathcal{X}^* \to \Delta^*$ family of elliptic curves $(X_s = E_s)$

$N = T - I$: $\alpha \longmapsto 0$

$\beta \longmapsto \alpha$

Basis for $\tilde{H}_{\mathbb{Q}}$: $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta - \frac{\log s}{2\pi i}\alpha$ $\quad \ell(s)$

Basis for $\ker(T-I)$: α .



$$(\bar{\mathcal{V}})_0 = (\hat{\mathcal{V}})_0 = \frac{H^1_{\lim}(X_s, \mathbb{C})}{F^1 H^1_{\lim} + \ker(T-I)_{\mathbb{Z}}}$$

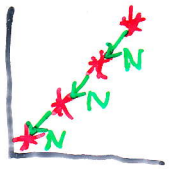
$$\tilde{\simeq} \ \mathbb{C}^* .$$

(II) $\mathcal{X}^* \to \Delta^*$ family of CY 3-folds w/ $\operatorname{rank} H^3(X_s) = 4$

Basis for H^1 : $\alpha, \beta, \gamma, \delta$

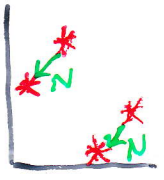
3 fundamentally different types of degeneration/LMHS are possible:

(i) $\alpha \overset{N}{\longmapsto} \beta \overset{N}{\longmapsto} \gamma \overset{N}{\longmapsto} \delta$



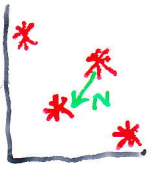
(ii) $\alpha \overset{N}{\longmapsto} \beta \overset{N}{\longmapsto} 0$, $\gamma \overset{N}{\longmapsto} \delta \overset{N}{\longmapsto} 0$

$$\tilde{\alpha} = \alpha - \ell(s)\beta \qquad \tilde{\gamma} = \gamma$$
$$\tilde{\gamma} = \gamma - \ell(s)\delta \qquad \tilde{\delta} = \delta$$
basis for $\tilde{H}_{\mathbb{Q}}$



(iii) $\alpha, \beta \overset{N}{\longmapsto} 0$

$\gamma \overset{N}{\longmapsto} \delta \overset{N}{\longmapsto} 0$



Examples (cont'd): place further $\boxed{\text{constraints}}$ on the VHS's.

(I) $\boxed{\mathcal{F}^\bullet \text{ is the nilpotent orbit of a trivial LMHS}}$

$$\mathcal{F}^1_s = \exp(\ell(s)N)\, F^1_{\lim} \qquad F^1_{\lim} = \mathbb{C}\langle\tilde{\beta}\rangle$$

$$\Rightarrow \hat{\mathcal{Q}}(=\bar{\mathcal{Q}}) = \frac{\mathcal{O}_\Delta\langle\tilde{\alpha},\tilde{\beta}\rangle}{\mathcal{O}_\Delta\langle\tilde{\beta}\rangle + \mathbb{Z}\langle\alpha,\beta\rangle} = \frac{\mathcal{O}_\Delta}{\mathbb{Z}\langle 1, \ell(s)\rangle}$$

with fibres $\mathbb{C}/\mathbb{Z}\langle 1,\ell(s)\rangle$ $(s\neq 0)$ and $\mathbb{C}/\mathbb{Z}$ $(s=0)$.

(II) $\boxed{\mathcal{F}^\bullet = \exp(\ell(s)N)\, F^\bullet_{\lim}}$ (nilpotent orbit)

where $\boxed{F^2_{\lim} = \mathbb{C}\langle\tilde{\alpha}-i\tilde{\gamma}, \tilde{\beta}-i\tilde{\delta}\rangle}$

Then
$$\bar{\mathcal{Q}} = \frac{\mathcal{O}_\Delta\langle\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}\rangle}{\mathcal{O}_\Delta\langle\tilde{\alpha}-i\tilde{\gamma},\tilde{\beta}-i\tilde{\delta}\rangle + \mathbb{Z}\langle\alpha,\beta,\gamma,\delta\rangle}$$

$$\cong \frac{\mathcal{O}_\Delta\langle\tilde{\gamma},\tilde{\delta}\rangle}{\mathbb{Z}\langle i\tilde{\gamma}+i\ell(s)\tilde{\delta},\ i\tilde{\delta},\ \tilde{\gamma}+\ell(s)\tilde{\delta},\ \tilde{\delta}\rangle}$$

fibres

$s\neq 0$: $\mathbb{C}^2 \Big/ \overbrace{\mathbb{Z}\left\langle \begin{pmatrix} i \\ i\ell(s)\end{pmatrix}, \begin{pmatrix} 0 \\ i\end{pmatrix}, \begin{pmatrix} 1 \\ \ell(s)\end{pmatrix}, \begin{pmatrix} 0 \\ 1\end{pmatrix}\right\rangle}^{\Gamma_s}$

$s=0$: $\mathbb{C}^2 \Big/ \mathbb{Z}\left\langle \begin{pmatrix} 0 \\ i\end{pmatrix}, \begin{pmatrix} 0 \\ 1\end{pmatrix}\right\rangle = \mathbb{C}\otimes\mathbb{C}/\mathbb{Z}[i] \supset \boxed{\mathbb{C}/\mathbb{Z}[i] = (\hat{\mathcal{G}})_0}$

(II, cont'd):

$\bar{Q}$ is non-Hausdorff: let $a \in \mathbb{Z}^{|a|}$, $b \in \mathbb{C}^*$, and consider the sequence of points $\in \mathcal{O}_\Delta(\tilde{\delta},\tilde{\delta})$

$$(s_n, \underline{v}_n) := \left(\exp\left(2\pi i\left(\frac{b+ni}{a}\right)\right), \underbrace{a\begin{pmatrix}1\\ l(s_n)\end{pmatrix} - n\begin{pmatrix}0\\ 1\end{pmatrix}}_{\begin{pmatrix}a\\ b\end{pmatrix}''} \in \Gamma_s\right) \xrightarrow{n\to\infty} \left(0, \begin{pmatrix}a\\ b\end{pmatrix}\right).$$

But in $\bar{Q}$, $(s_n, \underline{v}_n) \equiv (s_n, \underline{0})$ and so approaches both $\begin{pmatrix}0\\ 0\end{pmatrix}$ and $\neq \begin{pmatrix}a\\ b\end{pmatrix}$ in the quotient topology.

$\hat{Q}$ is Hausdorff: let $a = 0$, $b \in \mathbb{C}^{|\mathbb{Z}^{1,1}|}$. Then $\exists$

- $\epsilon > 0$ sufficiently small
- $M \gg 0$ sufficiently large

such that

$$\mathrm{Im}(z) > M \Rightarrow \left\| n_1\begin{pmatrix}i\\ il(s)\end{pmatrix} + n_2\begin{pmatrix}0\\ i\end{pmatrix} + n_3\begin{pmatrix}1\\ l(s)\end{pmatrix} + n_4\begin{pmatrix}0\\ 1\end{pmatrix} - \begin{pmatrix}0\\ b\end{pmatrix}\right\| > \epsilon$$

$(\forall\, \underline{n} \in \mathbb{Z}^4)$

This is not hard to check by hand.

Similar norm estimates lead to the next result:

Theorem (M. Saito): $\hat{Q}$ is always Hausdorff.

An admissible normal function (ANF) over Δ is a holomorphic section ν of $\mathcal{J}$ over Δ^*, some finite multiple $N\nu$ of which extends through $(\bar{\mathcal{J}})_0$, and satisfying the horizontality condition: any local lift $\tilde{\nu} \in \Gamma(U, \mathcal{H})$ satisfies $\nabla_{\partial/\partial t} \tilde{\nu} \in F^{m-1}$. By [GGK], the graph of $N\nu$ in fact lies in $\hat{\mathcal{J}}$. M. Saito's theorem thus leads to

Corollary 1: Assume $\nu \not\equiv 0$. Then the zero locus $Z(\nu) \subset \Delta$ of $\bar{\nu} \in \Gamma(\Delta, \hat{\mathcal{J}})$ does not have an accumulation point in Δ (i.e. is an analytic subvariety).

Corollary 2: Let $\nu \in \Gamma(S^*, \mathcal{J})$ be an ANF over a (smooth) [redacted] algebraic curve $S^*(/\mathbb{C})$. Then $Z(\nu) \subset S^*$ is an algebraic subvariety (i.e. a finite point set).

N.B. These results (Cor. 2 in particular) were also proved by Brosnan - Pearlstein, by a different method, and generalized to higher-dimensional S^*. Finally, two different constructions of Hausdorff Néron models generalizing GGK were done by Schnell and Kato-Nakayama-Usui, yielding more conceptual proofs.

Conjecture 1: Let $\mathbb{F} \subset \mathbb{C}$ be finitely generated $/\overline{\mathbb{Q}}$, $S^*/\mathbb{F}$ smooth curve, ν an ANF arising from a cycle $(\mathcal{X}^* \to S^*,\ \mathcal{Z} \in \mathcal{Z}^m(\mathcal{X})_{pr})$ defined $/\mathbb{F}$. Then $\mathcal{Z}(\nu)$ is an algebraic subvariety of S^* defined / algebraic extension of $\mathbb{F}$.

Remark: Bloch & Beilinson conjectured the existence of a decreasing filtration $F^\bullet_{BB}$ on the Chow groups $CH^m(X)$ of all smooth proj. varieties $X/\mathbb{C}$, satisfying various properties ($\Rightarrow$! if $\exists$).

Many filtrations satisfying most of these properties and conjecturally satisfying the remaining ones have been constructed; denote one of these by $\mathcal{I}^\bullet$. Now recall that one has invariants

(i) $cl : CH^m(X) \to H^{2m}(X, \mathbb{Q}) \cap F^m H^{2m}(X, \mathbb{C})$ fundamental class map

and

(ii) $AJ : \ker(cl) \to J^m(X)$ Abel–Jacobi map

Surjectivity of cl is the Hodge Conjecture (HC), to which we shall return. One has

$$\mathcal{I}^0 CH^m(X) = CH^m(X)$$
$$\mathcal{I}^1 CH^m(X) = \ker(cl)$$

Conjecture 2: $\boxed{\mathcal{I}^2 CH^m(X) = \ker(AJ).}$ $(\forall X$

Proposition [KP]: Conjectures 1 & 2 are equivalent.

III. Limits of normal functions & Arithmetic

Recall the semistable degeneration of threefolds (in affine form)

$$Y_s = \{\overbrace{s(x^5+y^5+z^5+w^5+1) - xyzw}^{F_s(x,y,z,w)} = 0\}, \quad s \in \Delta^*$$

with fibre

$$\tilde{Y}_0 = \bigcup_{i=1}^{5} \mathcal{W}_i \quad \left(\xrightarrow{\beta} Y_0 = \cup 5\, \mathbb{P}^3 / S \right)$$

over $\{0\}$. The component $\mathcal{W}_5$ > 4 Fermat-curve blowups $\tilde{B}_{j5}$ $(j=1,\dots,4)$. Denote by $\tilde{S}_5 (\subset \mathcal{W}_5)$ the proper transform of the standard Fermat quintic surface in $\mathbb{P}^3$.

The family $\{Y_s\}$ comes with a family of holomorphic 3-forms

$$\Omega^3(Y_s) \ni \omega(s) := \mathrm{Res}_{Y_s}\left(\frac{dx\wedge dy\wedge dz\wedge dw}{F_s(x,y,z,w)}\right).$$

We summarize how the example at the end of part (I) gets treated (and generalized).

Higher Chow groups

$$CH^p(X,n) := \frac{\left\{ \begin{array}{l} \text{"admissible, closed" codimension-}p \\ \text{algebraic cycles on } X \times \mathbb{A}^n \end{array} \right\}}{\text{"higher" rational equivalence}}$$

$$= H_n\left(\mathcal{Z}^p(X,\bullet), \partial\right)$$

were introduced by Bloch to compute algebraic K_n-groups of X, and come with regulator maps

$$reg^{p,n}: CH^p(X,n) \longrightarrow J^{p,n}(X)$$

$$\| $$

$$\frac{H^{2p-n-1}(X,\mathbb{C})}{F^p H^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Z})}$$

stated for $X/\bar{\mathbb{Q}}$; have implications for $X/\mathbb{Q}$

the arithmetic content of which is part of the Beilinson conjectures. The main point for us is:

$AJ_{\tilde{Y}_0}$ is built out of regulator maps on substrata

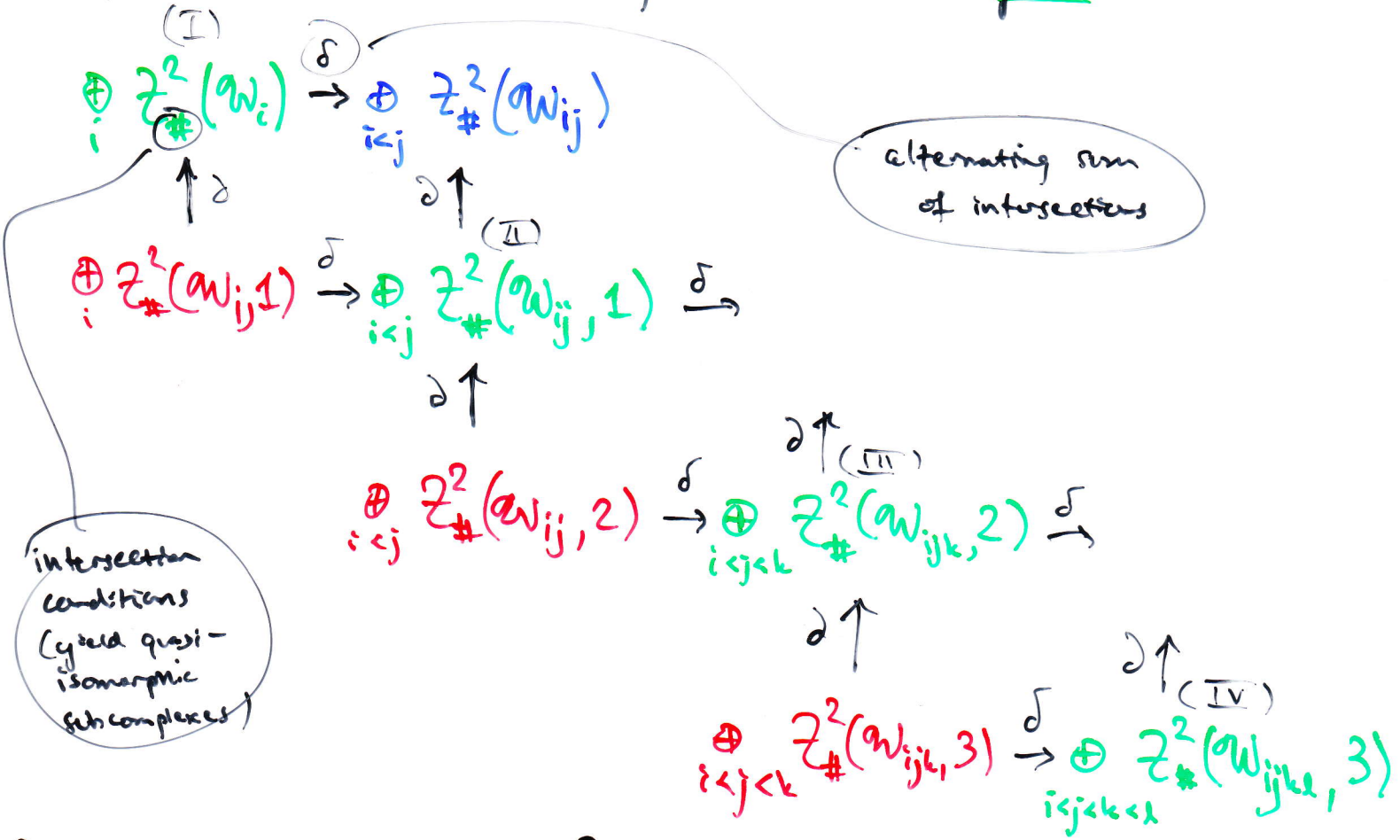
$$Gr^W_{-l} H^{2m}_M(\tilde{Y}_0, \mathbb{Z}(m)) \xrightarrow[Gr^m_{-l} AJ]{} Gr^W_{-l} J^m(\tilde{Y}_0)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\text{subquotient of } CH^m(\mathcal{W}^{[l]}, l) \xrightarrow{(reg^{m,l})} \text{subq. of } J^{m,l}(\mathcal{W}^{[l]})$$

We will focus here on the lowest "weight" piece.

Now, $H^4_M(\tilde{Y}_0, \mathbb{Q}(2))$ is computed by taking cohomology of the double complex "at the $\oplus$ of the terms in green"



Definition 1: $\mathcal{Z}_5 \in \mathcal{Z}^2(\mathcal{W}_5)_{hom}$ is **good** if it does not meet the $\mathcal{W}_{j5}$ $(j<5)$ and its intersections with the $\{\tilde{B}_{j5}\}$ are contained in $\tilde{\mathcal{S}}_5$.

Lemma 1: $\mathcal{Z}_5$ good $\Rightarrow$ $\mathcal{Z}_0 :=$

$\{(0,0,0,0,\mathcal{Z}_5), 0, 0, 0\}$ yields
(I) (II) (III) (IV)

a cocycle in the double complex deforming to an analytic family $\{\mathcal{Z}_s \in CH^2(Y_s)\}_{s \in \Delta^*}$.

Definition 2: $f, g \in \mathbb{C}(\mathbb{P}^1)^*$ is a good pair if (coord. u)

- $f \equiv 1$ on $|(g)| \cup \{0\}$
- $g \equiv 1$ on $|(f)| \cup \{\infty\}$
- poles of f are of order 2, except at $u = 1$ (allow order 3)

Writing $f(u) = \prod_j \left(1 - \frac{u}{\alpha_j}\right)^{m_j}$, $g(u) = \prod_k \left(1 - \frac{\beta_k}{u}\right)^{n_k}$, $\sum m_j = \sum n_k = 0$, we set

$$N_{f,g} := \sum_{j,k} m_j n_k \left(\frac{\beta_k}{\alpha_j}\right) \in \mathbb{Z}[\mathbb{C}^*].$$

Lemma 2: Let f, g be a good pair. Then (writing $\square = \mathbb{P}^1 \setminus \{1\}$)

$$V_{f,g} := \{(u, f(u), g(u)) \mid u \in \mathbb{P}^1\} \cap \square^3 \in Z^2(W_{2345}, 3).$$

(point)

By taking $\partial \pm \delta$ of cochains in the "real" part of the double complex, this $V_{f,g}$ can be moved from IV $\mapsto$ III $\mapsto$ II $\mapsto$ good cycle in I, given by

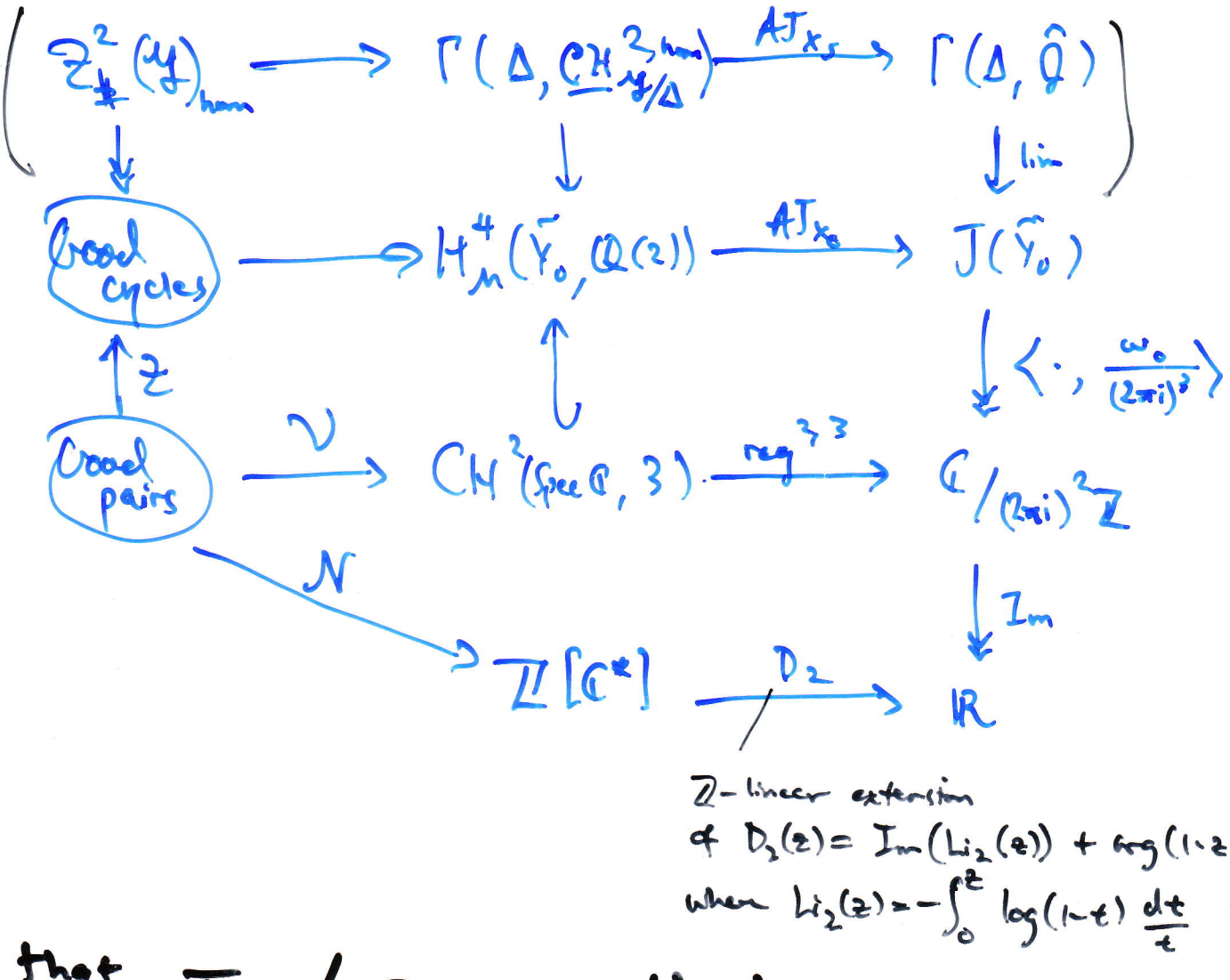
$$(Z_5)_{f,g} = \tilde{C}_{f,g} - \tilde{C}'_{f,g}$$

where (in $\mathbb{P}^3$)

$$C_{f,g} = \overline{\left\{(x,y,z) \in \mathbb{A}^3 \;\middle|\; \begin{array}{l}(1+x^5) f(-x^5) + y^5 = 0, \\ (1+x^5+y^5) g(-x^5) + z^5 = 0\end{array}\right\}}$$

$$- \overline{\left\{(x,y,z) \in \mathbb{A}^3 \;\middle|\; \begin{array}{l}(1+x^5) f(-x^5) + y^5 = 0, \\ 1 + x^5 + y^5 + z^5 = 0\end{array}\right\}}.$$

Theorem [GGK]: One has a diagram



$\mathbb{Z}$-linear extension of $D_2(z) = \mathrm{Im}(Li_2(z)) + \arg(1-z)\log|z|$, where $Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}$.

So that $$\mathrm{Im}\left\langle AJ_{X_0}(Z_0), \frac{\omega_0}{(2\pi i)^3}\right\rangle = D_2(\mathcal{N}_{f,g}).$$

More precisely, if $\{\Gamma_s\}_{s\in\Delta^*}$ is an arbitrary family of real 3-chains bounding on Z_s, then the truncated normal function $\int_{\Gamma_s}\omega(s)$ has

$\lim_{s\to s_0}\int_{\Gamma_s}\omega(s) \in \mathbb{C}/4\pi^2\mathbb{Z}$ is well-defined

with imaginary part $\sum_{j,k} m_j n_k D_2\left(\frac{\beta_k}{\alpha_j}\right)$.

This highlights the effects of arithmetic constraints (à la Beilinson) upon limits of ANF's of geometric origin (with no a priori arithmetic hypothesis).

IV. Singularities of normal functions

Earlier we talked about graphing a multiple of a normal function in $\hat{Q}$; this multiple gets rid of a torsion singularity. Of greater interest are nontorsion singularities, which only occur for ANFs over S of dimension > 1.

Theorem [Griffiths-Green]: The HC on a $2p$-dimensional (smooth projective) X is equivalent to the statement:

For all primitive Hodge (p,p)-classes ξ and very ample line bundles $\mathcal{L} \to X$, $\exists\, k \gg 0$ s.t. the 'natural' ANF $\nu_{\xi,k,\mathcal{L}}$ on $S' := \mathbb{P}H^0(X, \mathcal{L}^k)$ has a nontorsion singularity in the discriminant locus (dual variety) $\check{X} \subset S'$.

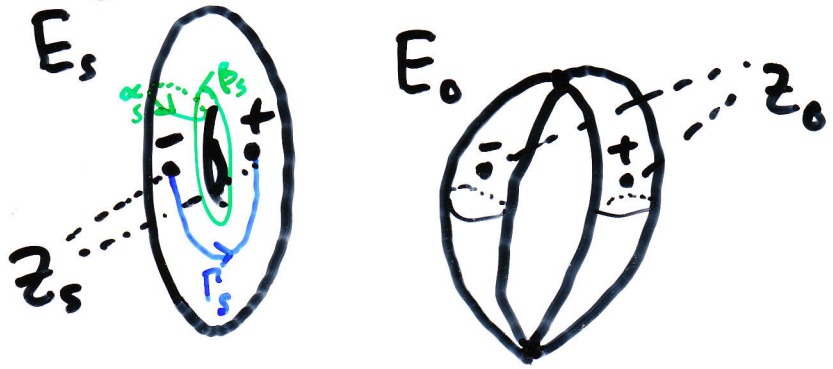
I will just try here to give an idea of what singularities are, for ANF's arising from families of divisors on families of elliptic curves. (The formal definition involves a mixed Hodge intersection cohomology complex.)

Torsion (1-parameter) example.

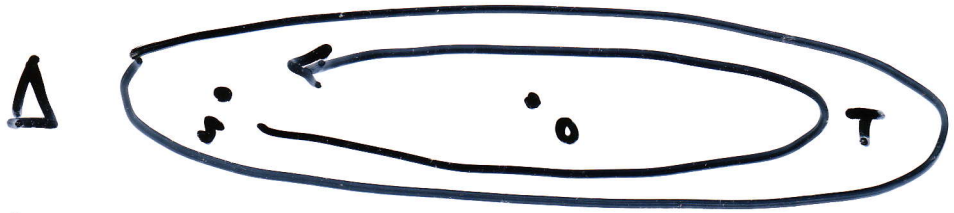
let $\xi \in \mathbb{C}$ be general + fixed.

$$C_s := \{x^2+y^2+s(x^2y^2+\xi)=0\}$$

defines a family of elliptic curves $\subset \mathbb{P}^1\times\mathbb{P}^1$ over Δ^* degenerating to a Néron 2-gon at $s=0$:



$$T:\ \alpha \mapsto \alpha$$
$$\beta \mapsto \beta + 2\alpha$$
$$\Gamma \mapsto \Gamma + \alpha$$



The cycle

$$Z_s := \left(i\sqrt{\tfrac{1+\xi s}{1+s}},\ 1\right) - \left(-i\sqrt{\tfrac{1+\xi s}{1+s}},\ 1\right) \underset{\text{hom}}{\equiv} 0$$

gives rise to the (nontorsion) normal function

$$\nu(s) := AJ_{C_s}(Z_s) = \int_{\Gamma_s}(\cdot) \in \Omega^1(C_s)^\vee \Big/ \int_{H_1(C_s,\mathbb{Z})}(\cdot) \cong J^1(C_s).$$

(which is independent of the choice of the family of paths Γ_s).

The finite group which measures the obstructions to ν extending through $\{0\}$ is

$$\mathrm{Sing}_0(\nu) := (T-I)\Gamma_s \in G := \frac{\mathrm{im}(T-I)_{\mathbb{Q}} \cap H^1(C_s,\mathbb{Z})}{\mathrm{im}(T-I)} \subset H^1(C_s,\mathbb{Z})$$

which here is

$$\bar{1} \in \mathbb{Z}/2\mathbb{Z}.$$

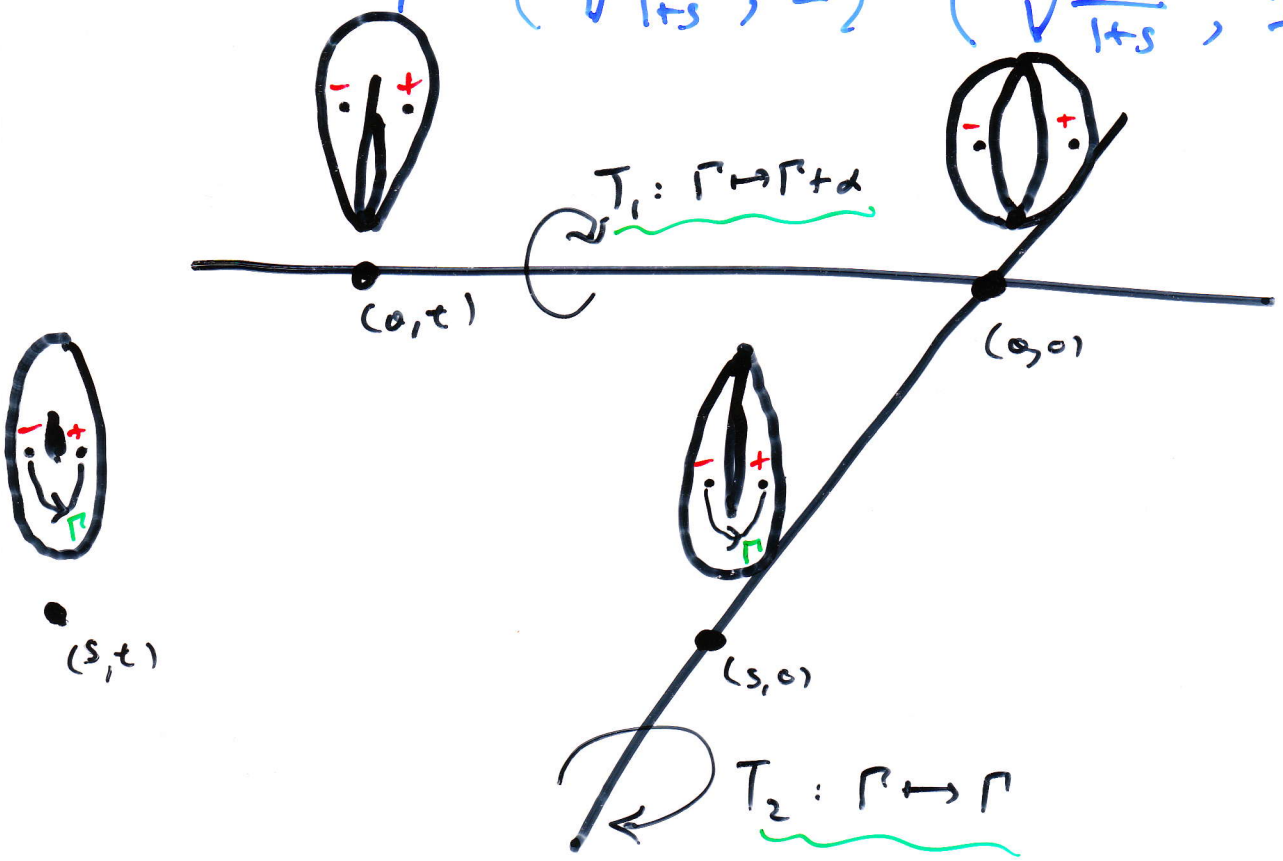
So $\Sigma\nu$ extends but ν doesn't. Note that

- $\otimes\mathbb{Q}$, we can "correct" this "problem": replace Γ_s by $\Gamma_s - \frac{1}{2}\beta_s$; this has no monodromy in cohomology
- alternatively, to avoid $\otimes\mathbb{Q}$, add a 2-torsion cycle like $T_s := (i\xi^{1/4}, \xi^{1/4}) - (-i\xi^{1/4}, -\xi^{1/4})$ to Σ_s.

Nontorsion (2-parameter) example: let ξ go free: consider

$$C_{s,t} := \{x^3y^2 + sx^2y^2 + t = 0\} \quad / \Delta^* \times \Delta^*,$$

and cycles $\Sigma_{s,t} := \left(i\sqrt{\frac{1+t}{1+s}}, 1\right) - \left(-i\sqrt{\frac{1+t}{1+s}}, 1\right)$:



3 ways to see what is "worse" here:

- $\otimes\mathbb{Q}$, try to correct monodromy: $\begin{cases} N_1(\Gamma) = \alpha, & N_1(\beta) = \alpha \\ N_2(\Gamma) = 0, & N_2(\beta) = \alpha \end{cases}$ impossible! $\Leftarrow$
- in the 2-torsion cycle T_s above, $\xi^{1/4}$ becomes $(t/s)^{1/4}$ — not well-defined!

- under the uniformization

$$\mathbb{P}^1/\{0,\infty\} \xrightarrow{\cong} C_{s,0}$$

$$z \longmapsto \left(\frac{2z}{1-sz^2}, \frac{2iz}{1+sz^2}\right)$$

$$\left[\frac{i}{s}\left(1+\sqrt{1+s}\right)\right] - \left[\frac{i}{s}\left(1-\sqrt{1+s}\right)\right] \longmapsto Z_{s,0}$$

the divisor corresponding to $Z_{s,0}$ moves, in the double-complex computing $H^2_M(\mathbb{P}^1/\{0,\infty\}, \mathbb{Q}(1))$, to $Z^1(\mathrm{Spec}(\mathbb{C}), 1) \twoheadrightarrow CH^1(\mathrm{Spec}\,\mathbb{C}, 1) \cong \mathbb{C}^*$

$$\frac{1+\sqrt{1+s}}{1-\sqrt{1+s}}$$

As $s \to 0$, this blows up — i.e. has a (pole) singularity in the sense of complex analysis.

Main points:

- $\exists$ of singularities is related to HC
- vanishing of singularity necessary to define limit (analogous to vanishing of $[Z]$ necessary to define $AJ(Z)$)
- singularities occur in codimension 2.

V. Higher normal functions and local mirror symmetry

A higher normal function is a holomorphic, horizontal section of a bundle of generalized Jacobians; they arise from families of higher Chow cycles.

Example: A family of $CH^2(E_s, 2) \cong K_2^{alg}(E_s)$-classes $\mathcal{Z}_s$ on a family of elliptic curves $/\mathbb{P}^1 \setminus \{0, \infty, \dots\} =: S^*$ gives a section of $\mathcal{J}^{2,2} \longrightarrow S^*$, where

$$J^{2,2}(E_s) \cong H^1(E_s, \mathbb{C}/\mathbb{Z}(1)), \quad s \in S^*.$$

This gives rise to two multivalued functions

$$\Psi_\alpha(s) := \int_{\alpha_s} \widetilde{reg^{2,2}(\mathcal{Z}_s)} \qquad \Psi_\beta(s) := \int_{\beta_s} \widetilde{reg^{2,2}(\mathcal{Z}_s)}$$

(lift class to $H^1(E_s, \mathbb{C})$)

Called regulator periods.

In local mirror symmetry, the VHS corresponding to a toric ally arising family of elliptic curves is "mirror" to a variation of Kähler structure on a certain open CY 3-fold X; this is expressed by the (inverse) mirror map

$$s \longrightarrow Q(s) = \underset{[DK]}{\exp}\left(\frac{\psi_\alpha(s)}{2\pi i}\right)$$

The local Gromov-Witten numbers N_D express the effective contribution made by an embedded toric Fano surface to "rational curves of degree D on X". One has the conjectural mirror theorem

$$\frac{n_0}{4\pi^2} + \sum_{D\geq 1} D^3 N_D Q^D = \frac{f(s)}{\left(\int_{\alpha_s} \omega(s)\right)^3}$$

- $f(s)$: Yukawa coupling — blows up at $s = s_0$
- $\int_{\alpha_s} \omega(s)$: abelian integral

which follows from the

Mirror Conjecture: the GW prepotential $= \psi_\beta(s(Q))$.
([DK], reformulating [CKYZ] + [Hosono])

Theorem [DK]: MC $\Longrightarrow \limsup_{D\to\infty} \sqrt[D]{|N_D|} = e^{-\frac{\mathrm{Im}(\psi_\alpha(s_0))}{2\pi}}$.

This gives a first view of a new interface between arithmetic and symplectic algebraic geometry, as $\mathrm{Im}(\psi_\alpha(s_0))$ is tied to special values of Dirichlet L-functions.

Major open ?'s: (1) What is the mirror of a higher NF?

(2) What is the mirror of an algebraic K-theory class?