Midterm, 2016 Spring, due March 22

(1) Calculate the Galois group of splitting field of $X^4 - 5$ over $\mathbb{Q}$. You must tell me what the group is, not just its order.

(2) Show that the ring $A = \mathbb{C}[x, y]/(f(x, y))$ where $f(x, y) = y^2 + 2xy + x^3 + x^2 + x$ is integrally closed. (You do not have to prove that f is irreducible.) Considering the Noether Normalization, $\mathbb{C}[x] \subset A$, find all $a \in \mathbb{C}$ such that the prime ideal $P_a = (x - a)\mathbb{C}[x]$ is ramified in A, which means the ramification index of at least one prime lying over P_a is greater than one.

(3) Let $f(x, y) = xy(x + y) \in \mathbb{F}_2[x, y]$, where $\mathbb{F}_2$ is the field of two elements. Show that for no change of variables, $u = ax+by, v = cx + dy$, with $a, b, c, d \in \mathbb{F}_2$ and $ad - bc \neq 0$, f can be made monic in u. (The reason why we assumed infinite field in our version of Noether Normalization).

(4) (a) Show that the maximal ideals of $A = \mathbb{C}[x, y]/(y^2 - x^3 + 1)$ are of the form $(x - a, y - b)$ with $b^2 = a^3 - 1$.
(b) Show that this ring is a Dedekind domain and the maximal ideal $(x - 1, y)$ is not principal.

(5) Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated module. Show that there exists a unique submodule $N \subset M$ such that either $N = 0$ or $\operatorname{Ass} N = \{\mathfrak{m}\}$ and $\mathfrak{m} \notin \operatorname{Ass} M/N$.

(6) Let A be a commutative ring containing $\mathbb{Q}$ and $D : A \to A$ a derivation. Assume that there exists an element $x \in A$ such that $D(x) = 1$ and given any element $a \in A$, $D^n(a) = 0$ for $n >> 0$. Let $B = \{a \in A | D(a) = 0\}$. Show that B is a subring of A and $A = B[x]$, a polynomial ring over B, for the above x.