Homework 9

(1) Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $k = A/\mathfrak{m}$ be the residue field. Show that a finitely generated module M is free if and only if $\mathrm{Tor}_1^A(M, k) = 0$ if and only if $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and for all modules N.

(2) Let $A = k[x, y]$, a polynomial ring in two variables over a field k and let $I = (x, y)$. Show that there is a natural surjection of rings $S(I) = \oplus_{n=0}^{\infty} S^n(I) \to \mathrm{Gr}(I) = \oplus_{n=0}^{\infty} I^n$. Show that the map $S^2(I) \to I$ is an isomorphism. (Last part requires some computation).

(3) If M is an A-module for a ring A and $M = N \oplus K$ for N, K A-modules, show that $T^n(M) = \oplus_{i=0}^{n} T^i(N) \otimes T^{n-i}(K)$, $S^n(M) = \oplus_{i=0}^{n} S^i(N) \otimes S^{n-i}(K)$ and similarly, $\Lambda^n(M) = \oplus_{i=0}^{n} \Lambda^i(N) \otimes \Lambda^{n-i} K$.

(4) Let A be a Noetherian ring and P a finitely generated projective module. Show that $S^n(P)$ and $\Lambda^n(P)$ are projective for all n.

(5) Let $A \subset B$ be rings and M a B-module. If B is a flat A-module and M is a flat B-module, show that M is a flat A-module.