Homework 8

(1) Let V be a vector space of dimension 2, with basis e_1, e_2 over a field k. Show that $V \otimes_k V$ is a vector space with basis $e_i \otimes e_j, 1 \leq i, j \leq 2$. If $\sum a_{ij} e_i \otimes e_j \in V \otimes V$ is decomposable, show that $a_{11}a_{22} = a_{12}a_{21}$ and thus not all elements of $V \otimes V$ are decomposable.

(2) Let $K \subset L$ be a finite extension of fields. Show that this extension is separable if and only if the ring $L \otimes_K \overline{K}$ has no non-trivial nilpotent elements.

(3) Let $\phi : A \to B$ be a ring homomorphism (as usual, rings are commutative with 1). Let M be an A-module and N be a B-module. So, N is also an A-module, using ϕ.
   (a) Show that $M \otimes_A B$ is a B-module via the action, $b(m \otimes c) = m \otimes bc$, for $m \in M, b, c \in B$.
   (b) Show that $\mathrm{Hom}_A(M, N)$ is a B-module via the action $(bf)(m) = bf(m)$ for $b \in B, f \in \mathrm{Hom}_A(M, N)$.
   (c) Show that $\mathrm{Hom}_B(M \otimes_A B, N)$ and $\mathrm{Hom}_A(M, N)$ are naturally isomorphic as B-modules.

(4) Let $A \to B$ be a ring homomorphism (as usual, rings are commutative with 1).
   (a) Show that there exists a natural surjective ring homomorphism $\phi : B \otimes_A B \to B$ with $\phi(x \otimes y) = xy$.
   (b) Show that the kernel of $\phi = I$ is generated as a $B \otimes_A B$ ideal by elements of the form $x \otimes 1 - 1 \otimes x, x \in B$.
   (c) Show that (easy) I/I^2 is a B module and there exists a well defined map $d : B \to I/I^2$, with $d(x) = x \otimes 1 - 1 \otimes x \in I/I^2$.
   (d) Show that d is an A-derivation.
   (e) If $D : B \to M$ is any A-derivation, where M is a B-module, show that there exists a B-module homomorphism $f : I/I^2 \to M$ such that $D = f \circ d$.
   (I/I^2 is called the module of Kähler differentials and denoted by $\Omega^1_{B/A}$.)

(5) If M is an A-module and $I \subset A$ is an ideal, show that $M \otimes_A A/I$ is naturally an A/I-module and isomorphic to M/IM.