Homework 7

(1) We say a commutative ring A is graded, if it can be written as $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, where A_0 is a ring, A_is are A_0 modules and the addition in A is the obvious one and $A_i A_j \subset A_{i+j}$ for all i, j.
   (a) Show that $A = k[x_1, \ldots, x_n]$, polynomial ring in n variables over a field k is a graded ring where A_d is the k vector space of all homogeneous polynomials of degree d.
   (b) Let A be a graded ring and let $D : A \to A$ be the map defined as $D((a_0, a_1, a_2, \ldots, a_n, \ldots)) = (0, a_1, 2a_2, \ldots, na_n, \ldots)$ is a derivation.

(2) A few problems on derivations in positive characteristic $p > 0$. All rings and fields will be of characteristic p.
   (a) Let $K \subset K(x) = L_1$ with $x \notin K$ and $x^p \in K$, be a field extension. Show that $\mathcal{D} = \mathrm{Der}_K(L_1)$ is a one dimensional L_1-vector space generated by $D = \frac{d}{dx}$. Calculate $(xD)^p$.
   (b) Prove a similar result for $L_n = K(x_1, \ldots, x_n)$, where $x_i^p \in K$, by first showing that we may choose n such that $p^n = [L_n : K]$. Again, calculate E^p, where E is the Euler derivation, $E = \sum x_i \frac{d}{dx_i}$.

(3) A few problems on modules and functors which often appear. Let A be a commutative ring and let $\mathcal{C}$ be the category of A-modules.
   (a) Let $X \in \mathcal{C}$ and define a map $F : \mathcal{C} \to \mathcal{C}$ by $M \mapsto \mathrm{Hom}_A(X, M)$. Show that this defines a functor.
   (b) If $0 \to M \to N \to P$ is an exact sequence in $\mathcal{C}$, show that the sequence, $0 \to F(M) \to F(N) \to F(P)$ is exact. Such functors are called *left exact*. Further show by an example that, even if we had surjectivity from $N \to P$ above, $F(N) \to F(P)$ may not be surjective.
   (c) Similarly, consider the map $G : \mathcal{C} \to \mathcal{C}$ given by $M \mapsto \mathrm{Hom}_A(M, X)$. Show that this too is a functor (to be precise, to the opposed category, since it will reverse arrows). As before, show that if we have an exact sequence $M \to N \to P \to 0$, we get an exact sequence $0 \to G(P) \to G(N) \to G(M)$.