Homework 6

(1) Let $f(x,y) \in \mathbb{C}[x,y]$ be an irreducible polynomial in x, y and let $A = \mathbb{C}[x,y]/f\mathbb{C}[x,y]$
   (a) Show that $\dim A = 1$.
   (b) By Hilbert Nullstellensatz, we have any non-zero prime ideal P of A is of the form $(x-a, y-b)$ where $a, b \in \mathbb{C}$ and $f(a,b) = 0$. Show that PA_P is generated by $x-a$ if $f_y = \frac{\partial f}{\partial y}(a,b) \neq 0$.
   (c) In the above situation, show that A_P is a dvr.
   (d) Deduce that A is integrally closed, if $f_x(a,b) \neq 0$ or $f_y(a,b) \neq 0$ whenever $f(a,b) = 0$.

(2) Let $R_n = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k of characteristic zero (for convenience) and let K_n be its fraction field. Let $F_n(t) \in K_n[t]$ be the polynomial, $F_n(t) = t^n + x_1 t^{n-1} + x_2 t^{n-2} + \cdots + x_{n-1}t + x_n$.
   (a) Show that $F_n(t)$ is irreducible for all $n \geq 1$.
   (b) Let L_n be its splitting field. Show that the Galois group of L_n over K_n is the symmetric group S_n.

(3) Let A be a complete local ring with maximal ideal M and containing $\mathbb{Q}$. Show that A contains a field k such that the composite $k \subset A \to A/M$ is an isomorphism, by the following steps.
   (a) Choose k to be the largest field in A, using Zorn's lemma.
   (b) If $k \to A/M$ is not an isomorphism, we distinguish two cases. If A/M is not algebraic, pick an element $\alpha \in A/M$, transcendental over k and let $t \in A$ be any lift of α. Show that $k(t) \subset A$, contradicting maximality.
   (c) Finally, assume A/M is algebraic over k and if not equal, pick $\alpha \in A/M - k$. Letting $f(u) \in k[u]$ to be its irreducible polynomial and t any lift of α, we have $f(t) = 0 \bmod M$. Apply Hensel's lemma.