Homework 5

(1) Let $0 \to F_1 \to F_0 \to M \to 0$ be an exact sequence of A-modules for a commutative ring A. Recall that this means $F_0 \to M$ is onto, image of $F_1 \to F_0$ is equal to the kernel of $F_0 \to M$ and $F_1 \to F_0$ is injective. Show that for any multiplicatively closed subset S, one has exact sequences, $0 \to S^{-1}F_1 \to S^{-1}F_0 \to S^{-1}M \to 0$ and for any ideal I, $F_1/IF_1 \to F_0/IF_0 \to M/IM \to 0$. For the latter you do not need to assume that $F_1 \to F_0$ is injective.

(2) Let A be a Noetherian ring and M a finitely generated module over A. Define the support of M as

$$\operatorname{Supp} M = \{P | P \text{ prime ideal}, M_P \neq 0\}.$$

We have defined in class the annihilator of M, $\operatorname{Ann} M = \{a \in A | aM = 0\}$, minimal primes and associated primes of M.

(a) Show that $\operatorname{Ass} M \subset \operatorname{Supp} M$.

(b) If $N \subset M$ is a submodule, show that $\operatorname{Supp} M = \operatorname{Supp} N \cup \operatorname{Supp}(M/N)$.

(c) Show that $P \in \operatorname{Supp} M$ if and only if $\operatorname{Ann} M \subset P$.

(3) Let A, M be as above and let P be a maximal ideal and let r be the dimension of M/PM as an A/P vector space. Show that there exists an $f \notin P$ such that for all maximal ideals Q not containing f, the dimension of M/QM as A/Q vector space is at most r.

(4) Let $A = k[x_1, \ldots, x_r]/I$ where k is a field, the maximal ideal generated by the all the variables has some power contained in I. Thus, A is an Artin local ring with the maximal P generated by the images of the variables, which by abuse of notation, we will still call x_i. Let M be a finitely generated module over A.

(a) If $\dim_k M = d$, show that for any chain of submodules $M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_p$ of M, $p \leq d$ and there is such a chain with $p = d$. Further show that for such a chain with $p = d$, $M_i/M_{i-1} \cong k = A/P$ for $i > 0$. As we defined in class, d is called the length of M and denoted by $\ell(M)$.

(b) If $N \subset M$, show that $\ell(M) = \ell(N) + \ell(M/N)$.

(c) Let $\omega_A = \operatorname{Hom}_k(A, k)$, the dualizing module. Show that the map $T : \operatorname{Hom}_A(M, \omega_A) \to \operatorname{Hom}_k(M, k)$ defined as, $T(f)(m) = f(m)(1)$ is an A-module isomorphism.

(d) Show that $\ell(\operatorname{Hom}_A(M, \omega_A)) = \ell(M)$. Deduce that the natural map $M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, \omega_A), \omega_A)$ is an isomorphism.