Homework 4

(1) Let $R = \mathbb{C}[x, y]$, polynomial ring in x, y. Show that $A = R/(y^2 - x^3)R$ is not integrally closed but $B = R/(y^2 - x^3 - x)R$ is. (You may assume that these are domains)

(2) Let A be a Noetherian integrally closed domain of finite Krull dimension, and let K be its fraction field. Prove that $A[t]$, polynomial ring in one variable over A is also integrally closed.

(3) Let $A = k[x, y]$, polynomial ring in two variables over a field k and let $M = A/(x^2, xy)$. What are the associated primes of M?

(4) Let M, N be A-modules and let $H = \mathrm{Hom}_A(M, N)$ be the set of all A-module homomorphisms from M to N. If we define for $f, g \in H$, $(f + g)(m) = f(m) + g(m)$, show that $f + g \in H$. Similarly, for $a \in A, f \in H$, define $(af)(m) = af(m)$ and show that $af \in H$. Show that H is (naturally) an A-module with these operations.

(5) We say that a module M over a ring A is finitely presented, if it is finitely generated and for some surjective homomorphism from a finitely generated free module $F \to M$ (which exists, since M is finitely generated), the kernel is also finitely generated. Show that if A is Noetherian, for a module to be finitely generated is same as it is finitely presented.

(6) Let A be a ring, M, N modules and assume that M is finitely presented. Show that for any multiplicatively closed subset $S \subset A$, $S^{-1}(\mathrm{Hom}_A(M, N))$ and $\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ are naturally isomorphic as $S^{-1}A$-modules.