Homework 3

(1) Let A be an integral domain. Show that A is integrally closed if and only if A_P (localization at the prime ideal P) is integrally closed for all primes P.
(2) If P is a prime ideal in a ring A, show that $\dim A_P \leq \dim A$.
(3) We say that a local domain A satisfies Serre condition S_2, if there exists two non-zero elements x, y in the maximal ideal such that if $ax = by$ for some $a, b \in A$, then $a = py, b = px$ for some $p \in A$. Show that if x, y is as above, and $ax^n = by^m$ for some $n, m \in \mathbb{N}$, then $a = py^m, b = px^n$ for some $p \in A$.
(4) A Noetherian integral domain of dimension one, which is integrally closed is called a *Dedekind domain*. We assume below that A is a Dedekind domain and K its fraction field. Also assume the following (which we shall prove later): For any $0 \neq a \in A$, there are only finitely many prime ideals containing a.
   (a) If $0 \neq I \subset K$ is a finitely generated A-module, define $I^{-1} = \{x \in K | xI \subset A\}$. Show that $II^{-1} = A$. (The notation II^{-1} as usual denotes the set of all finite sums of elements ab with $a \in I, b \in I^{-1}$). Deduce that I^{-1} is a finitely generated A-module. We call such a non-zero finitely generated A-submodule of K, a *fractionary ideal*.
   (b) Show that the set of fractionary ideals form an abelian group with multiplication as above.
   (c) If I is any fractionary ideal, show that there exists prime ideals $P_1, \ldots, P_n$ and $0 \neq e_i \in \mathbb{Z}$ (both unique) such that $I = P_1^{e_1} \cdots P_n^{e_n}$.
(5) Let A be a Noetherian integral domain of finite Krull dimension. Then A is integrally closed if and only if the following conditions are satisfied.
   (a) A_P is integrally closed for any prime ideal with $\dim A_P = 1$.
   (b) If $\dim A_P > 1$, then A_P satisfies S_2 condition above.

   This is called Serre's criterion for integrally closed rings.