Homework 2

(1) Let $0 \neq f \in k[[x_1, \ldots, x_n]]$ be a power series (and assume k is infinite for simplicity). Show that after a change of variables of the form $x_i + a_i x_n, i < n$, we may assume that $f(0, \ldots, 0, x_n) \neq 0$.

(2) Let $A = k[[x_1, \ldots, x_n]]/I$, where I is some proper ideal. Show that (after a change of variables) we have $k[[x_1, \ldots, x_d]] \subset A$ ($d \leq n$) and A is integral over the sub power series ring. (Hint: Use the above and Weierstrass preparation theorem.)

(3) Let K be a finite extension of $\mathbb{Q}$ and let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. We know that $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$ module from last week homework.
   (a) Show that $\mathcal{O}_K$ is a free module over $\mathbb{Z}$ (which is the same as a free abelian group) of rank $[K : \mathbb{Q}] = n$.
   (b) Let $\sigma_i, 1 \leq i \leq n$ be the distinct embeddings of K into $\mathbb{C}$ (over $\mathbb{Q}$, which is automatic). Show that the map $\phi : \mathcal{O}_K \to \mathbb{C}^n$, given by $a \mapsto (\sigma_1(a), \ldots, \sigma_n(a))$ is an injective group homomorphism and the image is a discrete subgroup of $\mathbb{C}^n$. (Recall that discrete means, given a ball B of radius $N > 0$ in $\mathbb{C}^n$, $B \cap \phi(\mathcal{O}_K)$ is a finite set.)
   (c) Let U be the group of units in $\mathcal{O}_K$ and consider the map, $\psi : U \to \mathbb{R}^n$, given by $u \mapsto (\log|\sigma_1(u)|, \ldots, \log|\sigma_n(u)|)$. Show that ψ is a group homomorphism, $\psi(U)$ is a discrete subgroup and the kernel of ψ is finite. Deduce that U is a finitely generated abelian group.