Homework 10

These are some exercises in Linear Algebra. Unless otherwise mentioned, we will assume we are working over an algebraically closed field K (often not necessary), all vector spaces are finite dimensional and tensor products etc. are over K. Many of the arguments will work with K replaced by an arbitrary commutative ring and vector spaces replaced by free modules (and sometimes, even projective modules).

(1) Let $f_i : V_i \to W_i$ be linear maps of finitely many vector spaces. Show that there is an induced linear map $\otimes f_i : \otimes V_i \to \otimes W_i$ with the property $\otimes f_i(\otimes v_i) = \otimes f_i(v_i)$ for $v_i \in V_i$.
(2) Let $f : V \to V$ be a linear map of a one dimensional vector space. Show that f is just multiplication by some $a \in K$. (Point here is that a is independent of the basis we may choose.)
(3) Let $f : V \to W$ be a linear map. The first problem shows that we have a linear map $\otimes^n f : T^n(V) \to T^n(W)$ for any n. Show that we also get induced linear maps $S^n(f) : S^n V \to S^n W$ and $\Lambda^n f : \Lambda^n V \to \Lambda^n W$ for all n. If f is injective (resp. surjective) show that the three induced maps above are injective (resp. surjective).
(4) Let $\dim V = n$ and let $f : V \to V$ be a linear map. This induces a map $\Lambda^n V \to \Lambda^n V$ and since this is a one dimensional vector space, from the second problem, this corresponds to an element $d \in K$. By choosing a basis for V and thus writing f as an $n \times n$ matrix, show that $d = \det f$. So, we have shown that $\det f$ is independent of the basis.
(5) For this problem, assume we are given a $f : V \to V$, linear map. We defined $P_f(X)$, the characteristic polynomial (which is monic of degree $\dim V$) and the minimal polynomial $Q_f(X)$ of f.
    (a) Show that $Q_f(X)$ divides $P_f(X)$ and they have the same roots. These roots are called the eigenvalues of f.
    (b) Let a be an eigenvalue of f and let $W = \{v \in V | f(v) = av\}$. Show that W is a non-zero subspace of V called the eigenspace corresponding to a.
    (c) Let a be an eigenvalue and let $W(a) = \{v \in V | (f - a \cdot \mathrm{Id})^N v = 0, \text{for some } N \in \mathbb{N}\}$. Show that $W(a)$ is a subspace of V and $V = \oplus W(a)$, where a ranges through the distinct eigenvalues of f.

(d) Let $a_1, \ldots, a_r$ be the eigenvalues (possibly some of them equal) of f, where $r = \dim V$. Calculate the eigenvalues of $T^n(f), S^n(f), \Lambda^n(f)$.