Homework 1

(1) Show that if $n > 1$ and $k \in \mathbb{Z}$, $|\Phi_n(k)| > k - 1$.
(2) Let D be a division ring (also called a skew field or non-commutative field) and let $Z(D) := \{a \in D | ax = xa \, \forall x \in D\}$ be the center.
   (a) Show that $Z(D)$ is a field and thus D is a vector space over $Z(D)$.
   (b) Assume that D is finite and let q be the cardinality of $Z(D)$ and $n = \dim_{Z(D)} D$. If $n > 1$, write the class equation for the finite group D^* (set of non-zero elements of D) and show that $\Phi_n(q)$ divides $q - 1$, leading to a contradiction. Thus deduce that $n = 1$ and D is a field.
(3) Let $K \subset \overline{K}$, a field and its algebraic closure. Let $S \subset \overline{K}$ be a subset with $S \cap K = \emptyset$. Show that there exists a field extension $K \subset E \subset \overline{K}$ such that $E \cap S = \emptyset$ and maximal with respect to this property, that is, if $E \subset E'$ is any larger extension then $E' \cap S \neq \emptyset$.
(4) Let $\mathbb{Q} \subset \overline{\mathbb{Q}}$ and let $\alpha \in \overline{\mathbb{Q}}$ be any irrational number. Let E be as in the previous problem, where $S = \{\alpha\}$. Show that any finite extension of E is cyclic, that is, it is Galois and the Galois group is cyclic.
(5) Let $n > 2$ be a positive integer and let $f(x) = x^n - 2$. If K is the splitting filed of f over $\mathbb{Q}$ and $n, \phi(n)$ are relatively prime, show that $[K : \mathbb{Q}] = n\phi(n)$.
(6) This is a set of facts we have already used in some fashion in the class. Let A be an integral domain, K its fraction field and L an algebraic extension of K.
   (a) If $b \in L$ show that there exists an $0 \neq a \in A$ such that ab is integral over A. Thus, if B is the integral closure of A in L, then the fraction field of B is L.
   (b) Assume L/K is a finite extension. If $b \in B$, show that $Tr_{L/K}(b), N_{L/K}(b) \in K$ are integral over A. So, if A is integrally closed (in its fraction field), then trace defines an A-module map $Tr_{L/K} : B \to A$ and norm defines a homomorphism of monoids, $B^* = B - \{0\} \to A^* = A - \{0\}$.
   (c) If A is integrally closed and Noetherian and L is a finite extension, show that B is a finitely generated A-module.
   (d) Describe the integral closures of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$, where d is any square free integer. (You should find out which elements are in this ring as explicitly as possible.)