ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 10.

In this lecture we finish the discussion of the Type Problem, one of the classical completeness problems stated in the first lecture. Recall that we denote by $\mathcal{E}_a = \mathcal{E}_{[0,a]}$ the family of exponential functions whose frequencies belong to the interval from 0 to a. If μ is a finite positive measure on $\mathbb{R}$ and $a > 0$ is a given number we would like to know if $\mathcal{E}_a$ is complete in $L^2(\mu)$, i.e. if finite linear combinations of its functions are dense in $L^2(\mu)$.

This problem appears in the work of Kolmogorov, Krein and Wiener in 1930-40's. As discussed in our previous lectures, it is related to other classical problems of analysis in several adjacent fields. Our goal in this lecture is to formulate a solution to the Type Problem found recently in [11].

Just like L^p problems on completeness of polynomials, the Type problem can be reformulated in terms of uniform Bernstein's approximation, using a proper version of Bakan's theorem discussed in lecture 7. Since such a reformulation will be important to us in the future, we recall the setup of Bernstein's approximation and Bakan's result.

We say that a function $W \geqslant 1$ on $\mathbb{R}$ is a *weight* if W is lower semi-continuous and $W \to \infty$ as $|x| \to \infty$. The weights are allowed to take infinite values.

The semi-norm $||f||_W$ is defined on the set of all continuous f such that $f/W \to 0$ at $\pm\infty$ as

$$||f||_W = \sup_{\mathbb{R}} \frac{|f|}{W}. \tag{0.1}$$

The following statement is an analog of Bakan's theorem [1] discussed in lecture 7 for the case of polynomials.

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Theorem 1. *Let $0 < p < \infty$ and $a > 0$ be constants, and let μ be a positive finite measure on $\mathbb{R}$. The family of exponentials $\mathcal{E}_a$ is complete in $L^p(\mu)$ if and only if μ can be represented as $\mu = W^{-p}\nu$ for some finite positive measure ν and a weight W such that $\mathcal{E}_a$ is complete in C_W.*

The proof is similar to the polynomial version (see exercises).

A condition for the completeness of $\mathcal{E}_a$ in C_W ($L^p(\mu)$) found in [11] relies on the notion of d-uniform sequences introduced in lecture 8 to treat the Gap Problem. Let us recall the definition of these sequences.

Let

$$... < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < ...$$

be a discrete sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $n \to \pm\infty$ and the sequence $\{I_n\}$ is short.

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of real points. We say that Λ is d-uniform if there exists a short partition $\{I_n\}$ such that

$$\Delta_n = d|I_n| + o(|I_n|) \quad \text{for all} \quad n \quad \text{(density condition)} \tag{0.2}$$

as $n \to \pm\infty$ and

$$\sum_n \frac{\Delta_n^2 \log|I_n| - E_n}{1 + \operatorname{dist}^2(0, I_n)} < \infty \quad \text{(energy condition)} \tag{0.3}$$

where Δ_n and E_n are defined as

$$\Delta_n = \#(\Lambda \cap I_n) \quad \text{and} \quad E_n = E(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n,\ \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

Via Bakan's theorem an equivalent version of the Type Problem can be formulated as follows: Given a weight W find

$$T_W = \inf\{a \mid \mathcal{E}_a \text{ is complete in } C_W\}.$$

As usual T_W is set to infinity if the set above is empty.

Here is an answer to that question (this is an equivalent version of the main result of [11]).

Theorem 2. *Let W be a weight on $\mathbb{R}$.*

$$T_W = \sup\left\{d \ : \ \exists d\text{-uniform } \Lambda,\ \sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} < \infty\right\}.$$

Returning to the L^p-approximation, recall that for the finite positive measure μ the quantities $\mathbf{G}^p_\mu$ were defined as

$$\mathbf{G}^p_\mu = \sup\{\ a \mid \exists\ f \in L^p(\mu), \int f(x)e^{2\pi i\lambda x}d\mu(x) = 0, \forall\ \lambda \in [0,a]\ \}. \tag{0.4}$$

Recall that the Type of μ can be equivalently defined as $T_\mu = \mathbf{G}^2_\mu$. Via Bakan's theorem, the last result can be reformulated as follows (see exercises). If μ is a finite positive measure on $\mathbb{R}$, we call a weight W μ-weight if

$$\int W d\mu < \infty.$$

Theorem 3. *Let μ be a finite positive measure on the line. Let $1 < p \leqslant \infty$ and $a > 0$ be constants.*

Then $\mathbf{G}^p_\mu \geqslant 2\pi a$ if and only if for any μ-weight W and any $0 < d < a$ there exists a d-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp}\mu$ such that

$$\sum \frac{\log W(\lambda_n)}{1+\lambda_n^2} < \infty. \tag{0.5}$$

Once again, the proof of theorem 3 is too long to include in this text. Let us discuss some of its corollaries and applications.

Discrete case.

The conditions of theorem 3 are simplified for many specific classes of measures. In particular, if the measure is discrete, or absolutely continuous with regular enough density, the weight W may be eliminated from the statement. Here we treat the discrete case that is important in spectral theory of differential operators and other adjacent areas. Our results in this section may be viewed as extensions of the result by Koosis mentioned earlier in the course.

The following statement gives a simplified formula for the type of a measure supported on a discrete sequence, excluding pathological cases when the counting function of the sequence grows exponentially.

Theorem 4. *Let $B = \{b_n\}$ be a discrete sequence of real points. Let*

$$\mu = \sum w(n)\delta_{b_n}$$

be a finite positive measure supported on B. Define

$$D = \sup \left\{ d \ : \ \exists \ d\text{-uniform } B' \subset B, \ \sum_{\lambda_n \in B'} \frac{\log w(n)}{1+n^2} > -\infty \right\}.$$

Then for any $1 < p \leqslant \infty$,

$$\mathbf{G}^p_\mu \geqslant 2\pi D.$$

If the counting function of B satisfies $\log(|n_B| + 1) \in L^1_\Pi$ then

$$\mathbf{G}^p_\mu = 2\pi D.$$

The condition $\log(|n_B| + 1) \in L^1_\Pi$ in the second part of the statement is sharp, see exercises.

In the case when the sequence is separated, the condition can be simplified even further. Note that for $p = 1$, $\mathbf{G}^1_\mu = 2\pi D_*(\Lambda)$ for any separated sequence Λ and any measure μ, $\operatorname{supp}\mu = \Lambda$, by the gap theorem from lecture 8. For $p > 1$ we have

Theorem 5. *Let $\Lambda = \{\lambda_n\}$ be a separated sequence and let*

$$\mu = \sum w(n)\delta_{\lambda_n}$$

be a finite positive measure supported on Λ. Define

$$D = \sup D_*(\Lambda'),$$

where the supremum is taken over all subsequences $\Lambda' \subset \Lambda$ satisfying

$$\sum_{\lambda_n \in \Lambda'} \frac{\log w(n)}{1+n^2} > -\infty. \tag{0.6}$$

Then

$$\mathbf{G}^p_\mu = 2\pi D$$

for all $1 < p \leqslant \infty$.

Proof. Suppose that $G^p_\mu > 2\pi D$ for some $D > 0, p > 1$. Define the μ-weight W as $W(\lambda_n) = (\mu(\{\lambda_n\})(1+\lambda_n^2))^{-1}$. Then by theorem 4 there exists a subsequence $\Lambda' \subset \Lambda$ such that $D_*(B') > D$ and (0.6) is satisfied.

In the opposite direction the statement follows directly from theorem 4 and the remark at the end of lecture 8. □

Finishing remarks.

The main goal of this course was to introduce the reader to the classical, yet still developing, branch of modern mathematics called the Uncertainty Principle in Harmonic Analysis. As we saw, this area is full of long standing problems, deep theorems and exciting applications. Despite almost a century of active research by many outstanding analysts, most fundamental problems in this field are wide open. What will the answer be if one replaces the interval with a union of intervals in the statement of the Beurling-Malliavin Problem? What if one replaces the interval with a more general set in the Type or the Hochstadt-Liberman Problems? These natural questions stand open, some for many decades, despite considerable efforts by the analytic community.

The Uncertainty Principle is definitely much wider than just the completeness and spectral problems discussed here. A natural next step after completeness are questions on excess and deficiency of exponential systems, polynomials or special functions, problems on frames, Riesz bases, etc. The reader interested in further material in this direction may consult the classic survey of Redheffer [8], a more modern survey by Khabibullin [5] (in Russian), books by Nikolski [7] and Seip [10].

For those readers interested in the spectral part of this course, the next natural step is to try to learn the Krein-de Branges theory of Hilbert Spaces of Entire functions, that translates spectral problems for differential operators into problems of complex and harmonic analysis. It generalizes the approach applied here to much broader class of differential operators, the so-called Krein canonical systems. The basics of that theory and further references can be found in [4] and [9]. The complex analytic part is covered in de Branges' book [3]. A modern account of the theory will (hopefully) soon appear in [6]. The Krein-de Branges theory experiences a new peak of popularity in recent years, partially due to the newfound connections to spectral problems, nonlinear Fourier tansform, number theory and the Riemann Hypothesis.

Exercises.

1) Prove Bakan's theorem for exponentials (first try to do it without looking at the polynomial proof).

2) Deduce theorem 2 from theorem 3.

3) Let $S \subset \mathbb{R}$ be a union of unit intervals. Define the measure μ as $d\mu(x) = \chi_S(x)/(1+x^2)$. Show that $\mathbf{G}^p_\mu$ is either 0 for all p or ∞ for all p.

4) Using theorem 3 or the results of [2] produce an example showing that the condition $\log(|n_B|+1) \in L^1_\Pi$ in the second part of the statement of the theorem 4 is sharp.

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