Lecture 3: Introduction to Calderón–Zygmund Operators

In this lecture we will start our study of Calderón–Zygmund operators in the one-parameter setting. The canonical example of such an operator is the Hilbert transform, which is given by

$$Hf(x) = \frac{p.v.}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

In the case of several variables, the canonical example becomes the Riesz transforms, which are given by

$$R_j f(x) = \frac{p.v.}{C(n)} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy \quad \forall j = 1, \ldots, n.$$

Note that these generate operators that are of convolution type, $T(f) = f * K$ for some appropriate kernel function K. However, the obvious estimates on the kernel give that

$$|K_j(x)| \lesssim \frac{1}{|x|^n}$$

and so these kernels are not integrable. However, it is easy to see that these kernels have an additional property of some "cancellation", which we will make more precise momentarily. These two properties together will imply that the operator are in fact bounded on $L^p(\mathbb{R}^n)$ when $1 < p < \infty$. Our goal in this lecture is to flesh out the details behind this fact.

1. Defintions of Calderón-Zygmund Operators

We will consider Calderón–Zygmund operators of the following form. We will have a convolution kernel $K(x)$ that satisfies the following conditions:

(a) (Size Condition) $|K(x)| \lesssim |x|^{-n}$;
(b) (Cancellation Condition) $\int_{r<|x|<R} K(x)dx = 0$ for all $0 < r < R < \infty$;
(c) (Hörmander Condition) $\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \lesssim 1$ when $|y| > 0$.

In some instances one can get away with the weaker hypothesis that $|\nabla K| \lesssim |x|^{-n-1}$ for condition (c). It is easy to see that the kernels for the Hilbert transform and the Riesz transforms satisfy these conditions. Our goal is to prove the following Theorem.

Theorem 1.1. *Suppose that the operator T given by*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

has a kernel K that satisfies conditions (a), (b) and (c) above. Then for $1 < p < \infty$ we have that $Tf : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with the operator norm controlled by the constants appearing in the definition of the kernel and the dimension.

Since the ideas that we need are contained in a weaker statement, we will also look at the following Theorem.

Theorem 1.2. *Suppose that the operator T given by*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

has a kernel K that satisfies conditions (a), (c) above. Suppose also that $|\hat{K}| \lesssim 1$. Then for $1 < p < \infty$ we have that $Tf : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with the operator norm controlled by the constants appearing in the definition of the kernel and the dimension.

In this statement of the theorem, $\hat{K}$ denotes the Fourier transform of the kernel K. By imposing the condition that $|\hat{K}| \lesssim 1$, we are supposing that the operator is in fact apriori bounded on $L^2(\mathbb{R}^n)$ as can easily be seen by Plancerel's Theorem.

While Theorem 1.1 is slightly more general than Theorem 1.2, they are essentially proved in the same manner. In fact, it is a good exercise to show that the conditions on the kernel imposed in Theorem 1.1 imply that for the function

$$K_\epsilon(x) = \begin{cases} K(x) & : \quad |x| \geq \epsilon \\ 0 & : \quad |x| < \epsilon. \end{cases}$$

that $\sup_x |\hat{K_\epsilon}| \lesssim 1$. One can also show that the kernels K_ϵ satisfy the conditions on the kernel in Theorem 1.1 (This originally contained a typo and referenced Theorem 1.2 instead of Theorem 1.1). So the method of proof behind Theorem 1.2 will contain all the main ideas necessary.

Exercise 1.3. *Verify that the kernel for the Hilbert transform satisfies these conditions.*

2. Weak-Type Estimates for Calderón–Zygmund Operators

In this section we prove a very useful decomposition Theorem for functions: The Calderón–Zygmund Decomposition.

Theorem 2.1. *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$ be given. Then there exists functions g and b such that*

(i) $f = g + b$;
(ii) $\|g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$ *and* $\|g\|_{L^\infty(\mathbb{R}^n)} \leq 2^n\lambda$;
(iii) $b = \sum_j b_j$ *were each b_j is supported on a dyadic cube Q_j. And, the collection of dyadic cubes $\{Q_j\}$ are disjoint;*
(iv) $\int_{Q_j} b_j(x)dx = 0$;
(v) $\|b_j\|_{L^1(\mathbb{R}^n)} \leq 2^{n+1}\lambda\,|Q_j|$;
(vi) $\sum_j |Q_j| \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}$.

The function g is called the "good function" since the properties that is has will imply that it belongs to all $L^p(\mathbb{R}^n)$ spaces. The function b is called the "bad function". It will turn out that it isn't such a bad function since it will have mean value zero by construction.

Proof. Start by considering dyadic cubes that satisify

$$|Q| \geq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}.$$

Sub-divide each of the cubes into 2^n children. Initialize two sets $\mathcal{S} = \emptyset$ and $\mathcal{Q}$ being the collection of dyadic cubes we start with. Now for each of these children, perform the following test:

$$\frac{1}{|Q|}\int_Q |f(x)|\,dx > \lambda \tag{2.1}$$

If a cube has this property, add it to the collection $\mathcal{S}$, and update $\mathcal{S} = \mathcal{S}\cup\{Q\}$ and $\mathcal{Q} = \mathcal{Q}\backslash Q$. For each cube in $\mathcal{Q}$ after this first test, if the cube doesn't have this property, then refine the cube by sub-division into 2^n children and, update $\mathcal{Q}$ to be the collection of these sub-cubes and then test each of these sub-cubes again. If the cubes now have the property (2.1) then we update $\mathcal{S}$. Repeat this partition process.

This produces a countable collection of cubes $\mathcal{S}$ such that for each cube $Q \in \mathcal{S}$ we have

$$|Q| \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}.$$

Note that the cubes are disjoint by construction. Since if a cube was selected, it was removed from further partitioning.

We can now define the parts of the functions f that will give the decomposition. For each cube $Q_j \in \mathcal{S}$ set

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y)dy\right)\chi_{Q_j}(x).$$

Then it is easy to see that $\int_{Q_j} b_j(x)dx = 0$ and that the support of each b_j is the cube Q_j. Let Q_j' denote the dyadic parent of the cube Q_j, i.e, the cube such that $Q_j \subset Q_j'$ and $|Q_j| = 2^{-n}\left|Q_j'\right|$. Note that if $Q_j \in \mathcal{S}$ than $Q_j' \notin \mathcal{S}$. So, for the cube Q_j' we have the opposite inequality to (2.1),

$$\frac{1}{\left|Q_j'\right|}\int_{Q_j'} |f(x)|\,dx \leq \lambda$$

Now using these observations we have,

$$\begin{aligned}
\|b_j\|_{L^1(\mathbb{R}^n)} &\leq 2\int_{Q_j} |f(x)|\,dx \\
&= 2\frac{\left|Q_j'\right|}{\left|Q_j'\right|}\int_{Q_j} |f(x)|\,dx \\
&\leq 2^{n+1}\,|Q_j|\,\frac{1}{\left|Q_j'\right|}\int_{Q_j'} |f(x)|\,dx \\
&\leq 2^{n+1}\,|Q_j|\,\lambda.
\end{aligned}$$

We then define $b = \sum_j b_j$ and $g = f - b$, which clearly gives that $f = g + b$. Looking a little closer at the definition of things, we see that

$$g(x) = \begin{cases} f(x) & : \quad x \in \mathbb{R}^n \setminus \left(\bigcup_j Q_j\right) \\ \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx & : \quad x \in Q_j. \end{cases}$$

By this definition of g we immediately see that $\|g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$. To see the L^∞ estimate, note that if $x \in Q_j$, then we have

$$|g(x)| = \left| \frac{1}{|Q_j|} \int_{Q_j} |f(x)|\, dx \right| \leq 2^n \frac{1}{|Q_j'|} \int_{Q_j'} |f(x)|\, dx \leq 2^n \lambda.$$

On the other hand, if $x \in \mathbb{R}^n \setminus \left(\bigcup_j Q_j \right)$, then there exists a sequence of non-selected cubes Q_k that converge to x. We then have that

$$\left| \frac{1}{|Q|_k} \int_{Q_k} f(x) dx \right| \leq \frac{1}{|Q_k|} \int_{Q_k} |f(x)|\, dx \leq \lambda.$$

By the Lebesgue Differentiation Theorem we then have that

$$|f(x)| \leq \lambda \quad x \in \mathbb{R}^n \setminus \left(\bigcup_j Q_j \right).$$

Combining these two estimates we see that $\|g\|_{L^\infty(\mathbb{R}^n)} \leq 2^n \lambda$. Finally, observe that

$$\begin{aligned} \sum_j |Q_j| &\leq \frac{1}{\lambda} \sum_j \int_{Q_j} |f(x)|\, dx \\ &= \frac{1}{\lambda} \int_{\bigcup_j Q_j} |f(x)|\, dx \\ &\leq \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

□

We now turn to show how to use this Theorem to deduce the following result.

Theorem 2.2. *Suppose that K is a Calderón–Zygmund kernel as defined above in Theorem 1.2. Then for all $f \in L^1(\mathbb{R}^n)$ and any $\lambda > 0$ we have*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Fix λ and $f \in L^1(\mathbb{R}^n)$. Apply the Calderón–Zygmund decomposition in Theorem 2.1 to obtain functions g, b so that $f = g + b$. Now observe that

$$\{x \in \mathbb{R}^n : |Tf| > \lambda\} \subset \left\{ x \in \mathbb{R}^n : |Tg| > \frac{\lambda}{2} \right\} \bigcup \left\{ x \in \mathbb{R}^n : |Tb| > \frac{\lambda}{2} \right\}$$

And so, we have

$$|\{x \in \mathbb{R}^n : |Tf| > \lambda\}| \leq \left| \left\{ x \in \mathbb{R}^n : |Tg| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |Tb| > \frac{\lambda}{2} \right\} \right|$$

and need to find estimates on each of these terms. The estimate on the good function is easy since we have

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^n : |Tg|>\frac{\lambda}{2}\right\}\right| &\lesssim \frac{\|Tg\|^2_{L^2(\mathbb{R}^n)}}{\lambda^2}\\ &\lesssim \frac{1}{\lambda^2}\|g\|^2_{L^2(\mathbb{R}^n)}\\ &\lesssim \frac{1}{\lambda^2}\lambda\|f\|_{L^1(\mathbb{R}^n)}\\ &= \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Here, we have used that $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded. In the final estimate we have used property (ii) in Theorem 2.1.

We now turn to understanding the estimate on the bad function. Let $\{Q_j\}$ be the cubes obtained in Theorem 2.1. Let Q_j^* denote the cube concentric with Q_j and having sidelength $2\sqrt{n}$ times the side length of Q_j. Then we have that

$$\left|\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right| = \left|\left(\bigcup Q_j^*\right)\cap\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right| + \left|\left(\bigcup Q_j^*\right)^c\cap\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right|.$$

Consider now the first term above, we then have that

$$\begin{aligned}\left|\bigcup Q_j^*\cap\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right| &\le \left|\bigcup Q_j^*\right|\\ &\le \sum_j |Q_j^*|\\ &\lesssim \sum_j |Q_j|\\ &\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

It only remains to handle the term

$$\left|\left(\bigcup Q_j^*\right)^c\cap\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right|$$

and for this one we will use the properties of the function b. Note that by simple estimates we have

$$\begin{aligned}\left|\left(\bigcup Q_j^*\right)^c\cap\left\{x\in\mathbb{R}^n : |Tb|>\frac{\lambda}{2}\right\}\right| &\lesssim \frac{1}{\lambda}\int_{\left(\bigcup Q_j^*\right)^c}|T(b)(x)|\,dx\\ &\le \frac{1}{\lambda}\sum_j\int_{(Q_j^*)^c}|Tb_j(x)|\,dx.\end{aligned}$$

Suppose for the moment that we proved

$$\int_{(Q_j^*)^c}|Tb_j(x)|\,dx \lesssim \int_{Q_j}|b_j(x)|\,dx \tag{2.2}$$

then we could continue the sum to find

$$\begin{aligned}\left|\left(\bigcup Q_j^*\right)^c \cap \left\{x \in \mathbb{R}^n : |Tb| > \frac{\lambda}{2}\right\}\right| &\lesssim \frac{1}{\lambda}\sum_j \int_{(Q_j^*)^c} |Tb_j(x)|\,dx \\ &\lesssim \frac{1}{\lambda}\sum_j \int_{Q_j} |b_j(x)|\,dx \\ &= \frac{1}{\lambda}\sum_j \|b_j\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_j |Q_j| \\ &\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

We now turn to proving (2.2). Here we will use the fact that b_j has mean value zero to introduce some cancellation into the integral. Let c_j denote the center of the cube Q_j. Observe that

$$\begin{aligned}\int_{(Q_j^*)^c} |T(b_j)(x)|\,dx &= \int_{(Q_j^*)^c} \left|\int_{Q_j} b_j(y)K(x-y)dy\right| dx \\ &= \int_{(Q_j^*)^c} \left|\int_{Q_j} b_j(y)(K(x-y)-K(x-c_j)dy\right| dx \\ &\leq \int_{Q_j} |b_j(y)| \left(\int_{(Q_j^*)^c} |K(x-y)-K(x-c_j)|\,dx\right) dy\end{aligned}$$

Focus on the inner integral now,

$$\int_{(Q_j^*)^c} |K(x-y)-K(x-c_j)|\,dx$$

and inspection reveals that this is very similar to what appears in condition (c) on the Calderón–Zygmund kernel. A change of variable, and simple estimates allow one to show

$$\int_{(Q_j^*)^c} |K(x-y)-K(x-c_j)|\,dx \leq \int_{|x|\geq 2|y-c_j|} |K(x-(y-c_j))-K(x)|\,dx \lesssim 1.$$

This then completes the proof of Theorem 2.2. □

With Theorem 2.2 at our disposal, it is very easy now to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. The hypothesis of the Theorem give that $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded. We have proved that in Theorem 2.2 that the operator $T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded too. Now, we apply the Marcinkiewicz Interpolation Theorem, Theorem 2.2 from Lecture 1, to conclude that $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ when $1 < p < 2$. To obtain the range $2 < p < \infty$, one simply considers the argument given, but now for the adjoint operator. It is easy to see that the kernel of the the adjoint will still be a Calderón–Zygmund kernel and so everything we have said so far applies again. □

3. Behavior Near L^1 and L^∞

As we have seen, the convolution-type Calderón-Zygmund operators are bounded on $L^p(\mathbb{R}^n)$ when $1 < p < \infty$. We have also see that the operators satisfy a weak-typ bound when $p = 1$. It turns out that we can have them be actually bounded if we change the target and domains.

Theorem 3.1. *Let T be a Calderón–Zygmund operator as defined above, then we have:*

$$T : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$T : L^\infty(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$$

While we haven't introduced the function spaces of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ (though we likely will), it is useful to at least have this theorem in mind.

References

The following are excellent books for an introduction to much of the harmonic analysis that we will be learning.

[1] Javier Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. ↑

[2] Loukas Grafakos, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009. ↑

[3] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. ↑

[4] ———, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. ↑