Lecture 9: Introduction to $\mathcal{D}$

We now turn to the main topic of this Internet Analysis Seminar: The Dirichlet Space. It turns out that almost all of the theory we have developed for the Hardy space $H^2(\mathbb{D})$ and $H^\infty(\mathbb{D})$ has a corresponding analogue in the case of the Dirichlet space. Over the next couple of weeks, we will explore these connections and similarities.

The topics that we will cover in this section of the Internet Analysis Seminar are the following:

- Carleson Measures for $\mathcal{D}$;
- The Corona Theorem for the Multiplier Algebra of $\mathcal{D}$;
- Characterization of the Interpolating Sequences of $\mathcal{D}$ and its multiplier algebra.

1. Basic Definitions for the Dirichlet Space $\mathcal{D}$

This first section is adapted from notes written by Richard Rochberg.

The Dirichlet space, $\mathcal{D}$, is the Hilbert space of functions $f \in \mathrm{Hol}\,(\mathbb{D})$ with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which

$$\|f\|_{\mathcal{D}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n\,|a_n|^2 = |f(0)|^2 + \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2\,dxdy < \infty. \tag{1.1}$$

Note that this condition insures that the radius of convergence of the power series is at least 1. The associated inner product between f and $g = \sum_{n=0}^{\infty} b_n z^n$ is given by

$$\langle f, g\rangle = \langle f, g\rangle_{\mathcal{D}} = a_0\overline{b_0} + \sum_{n=1}^{\infty} n a_n \overline{b_n} = f(0)\overline{g(0)} + \frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)}dxdy.$$

1.1. **The Dirichlet Space as a Reproducing Kernel Hilbert Space.** For $\zeta \in \mathbb{D}$ consider the map of $f \in \mathcal{D}$ to $f(\zeta)$. If we prove this is a continuous linear functional then the Riesz representation theorem will insure that there is a $k_\zeta \in \mathcal{D}$, the *reproducing kernel for* ζ, with the property that

$$f(\zeta) = \langle f, k_\zeta\rangle_{\mathcal{D}} \qquad \forall f \in \mathcal{D}. \tag{1.2}$$

Rather than prove this functional is continuous to obtain k_ζ; we will construct $k_\zeta \in \mathcal{D}$ so that (1.2) holds and conclude that the functional is continuous. What we do in the construction works *mutatis mutandis* for many other Hilbert spaces of holomorphic functions, such as what we saw in the case of $H^2(\mathbb{D})$

In fact there is a general class of Hilbert spaces where such considerations work, the reproducing kernel Hilbert spaces (RKHS's). An introduction to that theory is in [1].

Using (1.1) one checks that the polynomials are dense $\mathcal{D}$ and that the normalized monomials, $e_0(z) = 1$ and $e_n(z) = z^n/\sqrt{n}$, $n = 1, 2, \ldots$ are an orthonormal basis. Set

$$k_\zeta(z) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\zeta)}.$$

Because the numbers $e_n(\zeta)$ tend to zero rapidly with n we see that $k_\zeta \in \mathcal{D}$ and there is no problem justifying the following computation;

$$\begin{aligned}\langle f, k_\zeta\rangle &= \left\langle f(z), \sum e_n(z)\overline{e_n(\zeta)}\right\rangle \\ &= \sum \left\langle f(z), e_n(z)\overline{e_n(\zeta)}\right\rangle \\ &= \sum \langle f(z), e_n(z)\rangle\, e_n(\zeta) \\ &= \left(\sum \langle f, e_n\rangle\, e_n\right)(\zeta) \\ &= f(\zeta).\end{aligned}$$

Thus the reproducing kernel is

$$\begin{aligned}k_\zeta(z) &= \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\zeta)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n}(z\bar{\zeta})^n \\ &= 1 + \log\frac{1}{1-z\bar{\zeta}}. \qquad (1.3)\end{aligned}$$

The norm of the reproducing kernel is given by

$$\|k_\zeta\|^2 = \langle k_\zeta, k_\zeta\rangle = k_\zeta(\zeta) = 1 + \log\frac{1}{1-|\zeta|^2}.$$

It will be convenient to have a notation for the normalized reproducing kernels

$$\widehat{k_\zeta} = \frac{k_\zeta}{\|k_\zeta\|}.$$

Also, it is useful to note that for any $z, w \in \mathbb{D}$

$$\overline{k_z(w)} = k_w(z). \qquad (1.4)$$

Exercise 1.1. *Show that point evaluation in $\mathcal{D}$ is a bounded operation, namely for any $\lambda \in \mathbb{D}$*

$$|f(\lambda)| \le \|f\|_{\mathcal{D}}^2\, C(\lambda) \quad \forall f \in \mathcal{D}.$$

The constant $C(\lambda)$ should be the norm of the reproducing kernel k_λ.

1.2. **Defining Spaces by Presenting Kernel Functions.** We defined the Dirichlet space by giving a formula for the inner product. We then used the fact that the monomials were an orthogonal basis to obtain a formula for the reproducing kernel. However, one can also reverse the process and start with the kernel functions and use them to define the space.

Suppose we are given a set S, such as the disk, and a collection of "kernel functions" $k_x(y) = K(y,x)$ defined on $S \times S$. We suppose that $\forall x, y \in S$ we gave $K(y,x) = \overline{K(x,y)}$ and we suppose K is positive definite, which is to say that for any $\{x_i\}_{i=1}^n \subset S$, $\{a_i\}_{i=1}^n \subset \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} a_i\overline{a_j}K(x_j, x_i) \ge 0$$

with equality only if all the $a_j = 0$. Given such a K and hence the associated $\{k_x\}$ we define a function $\langle\cdot,\cdot\rangle$ on pairs of kernel functions on S by setting, for any $x, x' \in S$,

$$\langle k_x, k_{x'}\rangle = k_x(x'). \qquad (1.5)$$

We then extend the definition to finite linear combinations of the k_x by requiring that it be linear in the first variable and conjugate linear in the second. The space of finite linear combinations of the k_x with the function $\langle \cdot, \cdot \rangle$ is a pre-Hilbert space. Let H be the abstract completion of this pre-Hilbert space to a Hilbert space. Then H is the reproducing kernel Hilbert space generated by the kernel function K and the finite linear combinations of the kernel functions are dense in H. Although we have taken an abstract completion, we can regard any $F \in H$ as a function on S by defining, for $y \in S$, $F(y) = \langle F, k_y \rangle$. This agrees with the functional of evaluation at x if F is a finite linear combination of kernel functions.

Although this type of description for a space can be useful it can also hide a great deal. For instance, suppose S is the interval $(0, 1)$ and consider the RKHS H associated with the kernel function $K(x, y)$ defined on $S \times S$ by

$$K(x, y) = 1 + \log\left(\frac{1}{1 - xy}\right).$$

Tracing through the definitions we see that there is a natural isometric imbedding of H into $\mathcal{D}$. Next, note that a function in $\mathcal{D}$ that is orthogonal to the image of H in $\mathcal{D}$ must vanish on $(0, 1)$ and hence must be the zero function. Thus H is, or is naturally isometrically isomorphic to, $\mathcal{D}$. Thus functions in H, originally defined only on $(0, 1)$, all extend to be holomorphic on the unit disk; a fact not clear from the definition we gave for H.

1.3. **The Dirichlet Space, a Geometric Viewpoint.** Recall that the area of $f(\mathbb{D})$, the image of the unit disk under the map f with area counted according to multiplicity, can be computed using the *Dirichlet integral,* $\mathfrak{D}(f)$, defined by

$$\mathfrak{D}(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'|^2 \, dA = \sum_1^\infty n \, |a_n|^2 .$$

Precisely, $\text{Area}(f(\mathbb{D})) = \pi \mathfrak{D}(f)$. Comparing the expressions for $\mathfrak{D}(f)$ and $\|f\|_{\mathcal{D}}^2$ we have a geometric characterization of functions in the Dirichlet space; f is in the Dirichlet space exactly if $\text{Area}(f(\mathbb{D}))$ is finite. Using this characterization we can check that $\mathcal{D}$ neither contains nor is contained in $H^\infty(\mathbb{D})$, the space of bounded holomorphic functions on $\mathbb{D}$. An example of an unbounded function is $\mathcal{D}$ is the Riemann map of the unit disk onto any unbounded region of finite area, for instance $\{z = x + iy : 0 < y < \exp(-x^2)\}$. An example of a bounded function not in $\mathcal{D}$ is given by any infinite Blaschke product, such a function maps the disk to an infinite sheeted cover of the disk. Alternatively one can give an elementary example:

$$g(z) = \exp\left(i \log \frac{1 + z}{1 - z}\right). \tag{1.6}$$

The map of z to $(1 + z)/(1 - z)$ takes the disk to the right halfplane. The logarithm takes the right half plane to the horizontal strip $\{z = x + iy : |y| < \pi/2\}$. Multiplication by i then produces the vertical strip with $-\pi/2 < x < \pi/2$ which is then mapped in an infinite-valent manner to the ring

$$\{z : \exp(-\pi/2) < |z| < \exp(\pi/2)\}.$$

Thus g is bounded but, because $\pi \mathfrak{D}(g)$ computes the area of the image with multiplicity, $\mathfrak{D}(g) = \infty$, and hence g is not in the Dirichlet space.

Sometimes it is convenient to measure the size of functions in the Dirichlet space by $\mathfrak{D}(f)^{1/2}$, without involving the value $f(0)$. Common ways to do this are to note that $\mathfrak{D}(f)^{1/2}$

is a norm on $\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}$, or that it is a norm on equivalence classes of functions modulo constants, or to just accept the fact that, on $\mathcal{D}$, $\mathfrak{D}(f)^{1/2}$ is only a quasinorm. Focusing on $\mathfrak{D}(f)^{1/2}$ highlights a fundamental invariance property of the Dirichlet space. Suppose ϕ is a conformal automorphism of the disk. Then, by the area interpretation, or by change of variables in the integral, $\mathfrak{D}(f) = \mathfrak{D}(f \circ \phi)$ and hence the map of $f(z)$ to $f(\phi(z)) - f(\phi(0))$ is a unitary map of $\mathcal{D}_0$. This fact is sometimes referred to, a bit imprecisely, by saying that the Dirichlet space is conformally invariant.

Exercise 1.2. *Show that $\mathfrak{D}(f) = \mathfrak{D}(f \circ \phi)$ and hence the map of $f(z)$ to $f(\phi(z)) - f(\phi(0))$ is a unitary map of $\mathcal{D}_0$.*

1.4. **A Slightly Different Norm for the Dirichlet Space.** In some contexts a slightly different norm and inner product for the Dirichlet space is used. A norm equivalent to the one we defined is given by, for $f(z) = \sum_0^\infty a_n z^n \in \mathcal{D}$, setting

$$|||f|||_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)\,|a_n|^2 < \infty. \tag{1.7}$$

The associated inner product is then given, for $g(z) = \sum_{n=0}^\infty b_n z^n \in \mathcal{D}$, by

$$\langle f, g\rangle_{\mathcal{D}} = \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n}$$

and the associated reproducing kernel function is then

$$k_\zeta(z) = \frac{1}{\bar{\zeta} z} \log\left(\frac{1}{1 - \bar{\zeta} z}\right).$$

1.5. **The Choice of Norm for the Dirichlet Space.** The functionals $\|\cdot\|_{\mathcal{D}}$ and $|||\cdot|||_{\mathcal{D}}$ are equivalent Hilbert space norms on the linear space $\mathcal{D}$. Hence for many questions, for instance the question of boundedness or of invertibility of a linear operator, the choice of particular norm makes little or no difference. However there are contexts which we will be considering in which the choice matters more.

We noted earlier that the norm $\|\cdot\|_{\mathcal{D}}$ and the functional $\mathfrak{D}(f)$ connect questions on the Dirichlet space with issues in geometric function theory.

It is also possible to rewrite the formula of $\mathfrak{D}(f)$ in ways that are more complicated but which give very subtle detailed insight into the relation between local behavior f and the value of $\mathfrak{D}(f)$. This work, from a classical formula of Douglas for $\mathfrak{D}(f)$ through the Sundberg-Richter expression for the local Dirichlet integral.

On the other hand, one of the fundamental facts about the Dirichlet space normed by $|||\cdot|||_{\mathcal{D}}$ is that it has the "Pick property". We discuss this property and some of its consequences, including a very elegant and useful relationship between the space and its "multiplier algebra".

The Pick property, which is shared by the Hardy space, is not an isomorphism invariant. It is a property that holds for the Hilbert space $(\mathcal{D}, |||\cdot|||_{\mathcal{D}})$ but not for the Hilbert space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$.

In these notes we will move from one norm to the other. In most cases the choice is one of convenience or elegance. However in certain instances we will have to use the special norm for particular results to hold.

2. The Multiplier Algebra for $\mathcal{D}$

We now turn to studying the multiplier algebra for $\mathcal{D}$. First, recall that the multiplier algebra of a space $H^2(\mathbb{D})$ is the collection of functions such that

$$\mathcal{M}_{H^2(\mathbb{D})} := \{\varphi \in \text{Hol}(\mathbb{D}) : \|\varphi f\|_{H^2(\mathbb{D})} \leq C \|f\|_{H^2(\mathbb{D})} \quad \forall f \in H^2(\mathbb{D})\}$$

We can place a norm on this space by defining

$$\|\varphi\|_{\mathcal{M}_{H^2}} := \inf\{C : \|\varphi f\|_{H^2(\mathbb{D})} \leq C \|f\|_{H^2(\mathbb{D})} \quad \forall f \in H^2(\mathbb{D})\}.$$

We now repeat this idea, but in the case of the Dirichlet space. The multiplier algebra of a space $\mathcal{D}(\mathbb{D})$ is the collection of functions such that

$$\mathcal{M}_{\mathcal{D}(\mathbb{D})} := \{\varphi \in \text{Hol}(\mathbb{D}) : \|\varphi f\|_{\mathcal{D}(\mathbb{D})} \leq C \|f\|_{\mathcal{D}(\mathbb{D})} \quad \forall f \in \mathcal{D}(\mathbb{D})\}$$

This space is then normed by

$$\|\varphi\|_{\mathcal{M}_{\mathcal{D}}} := \inf\{C : \|\varphi f\|_{\mathcal{D}(\mathbb{D})} \leq C \|f\|_{\mathcal{D}(\mathbb{D})} \quad \forall f \in \mathcal{D}(\mathbb{D})\}.$$

Exercise 2.1. *Show that with this definition that $\mathcal{M}_{\mathcal{D}(\mathbb{D})}$ is a Banach algebra.*

Unfortunately, we no longer have the case that $\mathcal{M}_{\mathcal{D}(\mathbb{D})} = H^\infty(\mathbb{D})$. However we do have the following being true.

Proposition 2.2.

$$\mathcal{M}_{\mathcal{D}(\mathbb{D})} \subsetneq H^\infty(\mathbb{D})$$

with

$$\|\varphi\|_{H^\infty(\mathbb{D})} \leq \|\varphi\|_{\mathcal{M}_{\mathcal{D}}}.$$

Proof. Let $\varphi \in \mathcal{M}_{\mathcal{D}}$ and let M_φ be the operator of multiplication by φ acting on $\mathcal{D}(\mathbb{D})$. In this notation we then have that $\|M_\varphi\| = \|\varphi\|_{\mathcal{M}_{\mathcal{D}}}$. Let M_φ^* be the adjoint of the operator M_φ. We select $\zeta, z \in \mathbb{D}$ and compute

$$\begin{aligned} M_\varphi^* k_\zeta(z) &= \left\langle M_\varphi^* k_\zeta, k_z \right\rangle \\ &= \langle k_\zeta, M_\varphi k_z \rangle_{\mathcal{D}(\mathbb{D})} \\ &= \langle k_\zeta, \varphi k_z \rangle_{\mathcal{D}(\mathbb{D})} \\ &= \overline{\langle \varphi k_z, k_\zeta \rangle}_{\mathcal{D}(\mathbb{D})} \\ &= \overline{\varphi(\zeta) k_z(\zeta)} \\ &= \overline{\varphi(\zeta)} k_\zeta(z). \end{aligned}$$

Here k_ζ is the reproducing kernel for the Dirichlet space $\mathcal{D}$. Thus, k_ζ is an eigenvector of M_φ^*, the adjoint of the multiplication operator, with eigenvalue $\overline{\varphi(\zeta)}$. Hence, $|\varphi(\zeta)| \leq \left\|M_\varphi^*\right\| = \|M_\varphi\|$. Taking the supremum over $\zeta \in \mathbb{D}$ gives the desired estimate.

We next provide an example of a function that is in $H^\infty(\mathbb{D})$, but can not be in $\mathcal{M}_{\mathcal{D}}$. Note that since $1 \in \mathcal{D}$ for any $\varphi \in \mathcal{D}_{\mathcal{D}}$ we must have that $\varphi \in \mathcal{D}$. Consider the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n^3}}{n^2}.$$

Note that we have $f \in H^\infty(\mathbb{D})$ since

$$|f(z)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

However, we don't have $f \in \mathcal{D}$. TO see this, note that

$$f'(z) = \sum_{n=1}^{\infty} \frac{n^3 z^{n^3-1}}{n^2} = \sum_{n=1}^{\infty} n z^{n^3-1},$$

but this clearly has infinite norm in $\mathcal{D}$. □

Exercise 2.3. *Show that for the Besov-Sobolev space $B_2^\sigma(\mathbb{D})$ with $0 \leq \sigma < \frac{1}{2}$ given by*

$$\|f\|_{B_2^\sigma}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\sigma} dA(z)$$

that

(a) *$\mathcal{M}_{B_2^\sigma}$ is a Banach algebra;*
(b) *$\mathcal{M}_{B_2^\sigma} \subsetneq H^\infty(\mathbb{D})$.*

While the observation that $\mathcal{M}_{\mathcal{D}} \subsetneq H^\infty(\mathbb{D})$ is useful, it would be better to have a full characterization of this algebra in terms of objects that we already have seen. This is the content of the next proposition.

First, we define the space $\mathcal{X}$ to be the collection of functions such that $|b'|^2 dA(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}$. Namely, a function $b \in \mathcal{X}$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2 |b'(z)|^2 dA(z) \leq C \|f\|_{\mathcal{D}}^2 \quad \forall f \in \mathcal{D}.$$

We can norm this space $\mathcal{X}$ by setting $d\mu_b := |b'(z)|^2 dA(z)$ and

$$\|b\|_{\mathcal{X}}^2 = |b(0)|^2 + \inf\left\{C : \int_{\mathbb{D}} |f(z)|^2 d\mu_b(z) \leq C \|f\|_{\mathcal{D}}^2\right\}.$$

By the geometric characterization of Carleson measures we have already seen, we have another (equivalent) view point of the space $\mathcal{X}$. With this space we have the following proposition.

Proposition 2.4.

$$\mathcal{M}_{\mathcal{D}} = H^\infty(\mathbb{D}) \cap \mathcal{X}$$

Moreover,

$$\|b\|_{\mathcal{M}_{\mathcal{D}}} \approx \|b\|_\infty + \|b\|_{\mathcal{X}}$$

Proof. First suppose that $b \in H^\infty(\mathbb{D}) \cap \mathcal{X}$. We will show that this function is then a multiplier on $\mathcal{D}$. Let $f \in \mathcal{D}$. Then we have

$$\begin{aligned}
\|bf\|_{\mathcal{D}}^2 &= |b(0)|^2 |f(0)|^2 + \int_{\mathbb{D}} |(bf)'(z)|^2 dA(z) \\
&\leq \|b\|_\infty^2 \|f\|_{\mathcal{D}}^2 + \int_{\mathbb{D}} |b'(z)|^2 |f(z)|^2 dA(z) + \int_{\mathbb{D}} |b(z)|^2 |f'(z)|^2 dA(z) \\
&\leq 2\|b\|_\infty^2 \|f\|_{\mathcal{D}}^2 + \|b\|_{\mathcal{X}}^2 \|f\|_{\mathcal{D}}^2.
\end{aligned}$$

This then implies that

$$\|b\|_{\mathcal{M}_\mathcal{D}} \leq \sqrt{2}\,\|b\|_\infty + \|b\|_\mathcal{X} \lesssim \|b\|_\infty + \|b\|_\mathcal{X}\,.$$

In the other direction, suppose that we know $b \in \mathcal{M}_\mathcal{D}$. First, observe that by our previous computations we have that $b \in H^\infty(\mathbb{D})$ and $\|b\|_\infty \leq \|b\|_{\mathcal{M}_\mathcal{D}}$. So it only remains to show that we have $\|b\|_X \leq \|b\|_{\mathcal{M}_\mathcal{D}}$. Again, observe that for $f \in \mathcal{D}$ we have

$$\begin{aligned}\int_\mathbb{D} |f(z)|^2\,|b'(z)|^2\,dA(z) &= \int_\mathbb{D} |(bf)'(z) - b(z)f'(z)|^2\,dA(z)\\ &\leq \|bf\|_\mathcal{D}^2 + \int_\mathbb{D} |b(z)|^2\,|f'(z)|^2\,dA(z)\\ &\leq \|b\|_{\mathcal{M}_\mathcal{D}}^2\,\|f\|_\mathcal{D}^2 + \|b\|_\infty^2\,\|f\|_\mathcal{D}^2\\ &\leq 2\,\|b\|_{\mathcal{M}_\mathcal{D}}^2\,\|f\|_\mathcal{D}^2\,.\end{aligned}$$

This implies that the norm of the Carleson measure $d\mu_b(z) = |b'(z)|^2\,dA(z)$ is controlled by $\sqrt{2}\,\|b\|_{\mathcal{M}_\mathcal{D}}$. Also, we have that $|b(0)| \leq \|b\|_{\mathcal{M}_\mathcal{D}}$ and so this gives

$$\|b\|_\mathcal{X} \leq \sqrt{3}\,\|b\|_{\mathcal{M}_\mathcal{D}}\,.$$

And, so this then gives that

$$\|b\|_\infty + \|b\|_\mathcal{X} \lesssim \|b\|_{\mathcal{M}_\mathcal{D}}\,.$$

□

Exercise 2.5. *For the Besov-Sobolev space $B_2^\sigma(\mathbb{D})$ with $0 \leq \sigma < \frac{1}{2}$ given by*

$$\|f\|_{B_2^\sigma}^2 = |f(0)|^2 + \int_\mathbb{D} |f'(z)|^2\,(1-|z|^2)^{2\sigma} dA(z)$$

(a) *Define the space $\mathcal{X}_\sigma$ in an analogous manner as what appears above;*
(b) *Show that $\mathcal{M}_{B_2^\sigma} = \mathcal{X}_\sigma \cap H^\infty(\mathbb{D})$.*

References

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. ↑1