Lecture 4: Examples of Carleson Measures and the Multiplier Algebra of $H^2(\mathbb{D})$

Having now provided the characterization of the Carleson measures for $H^2(\mathbb{D})$, it is useful to collect numerous examples of these measures so we have in mind what these measures "look" like. After we study these examples, we will look at an abstract way to study the algebra $H^\infty(\mathbb{D})$, which will serve as motivation when we study the Dirichlet space later on.

1. Examples of Carleson Measures for $H^2(\mathbb{D})$

We now want to collect a couple of different families of Carleson measures that frequently appear. The first is a well known lemma due to Uchiyama (see the proof in [2]).

Lemma 1.1 (Uchiyama's Lemma)**.** *Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(\mathbb{D})$*

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\,|f(z)|^2\,d\mu(z) \leq e\|\varphi\|_\infty \|f\|_2^2.$$

Here $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dA(z)$, and $\widetilde{\Delta} = \frac{1}{4}\Delta = \partial\overline{\partial}$.

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\widetilde{\Delta}\big(e^{\varphi(z)}\,|f(z)|^2\big) = e^{\varphi}\widetilde{\Delta}\varphi\,|f|^2 + e^{\varphi}\,|\partial\varphi f + \partial f|^2 \geq \widetilde{\Delta}\varphi\,|f|^2\,.$$

Then Green's formula implies

$$\begin{aligned}\int_{\mathbb{D}} \widetilde{\Delta}\varphi\;|f(z)|^2\;d\mu(z) &\leq \int_{\mathbb{D}} \widetilde{\Delta}\big(e^{\varphi}\,|f|^2\big)\,d\mu(z)\\ &= \int_{\mathbb{T}} e^{\varphi(\xi)}\,|f(\xi)|^2\;dm(\xi) - e^{\varphi(0)}\,|f(0)|^2\\ &\leq e\int_{\mathbb{T}} |f(\xi)|^2\;dm(\xi) = e\;\|f\|_{H^2}^2\,.\end{aligned}$$

□

Remark 1.2. It is easy to see, that the above Lemma implies the embedding

$$\int_{\mathbb{D}} |f(z)|^2\;d\mu(z) \leq C\int_{\mathbb{T}} |f(\xi)|^2\;dm(\xi)$$

(with $C = e$) for all analytic functions f. Using the function $4/(2-\varphi)$ instead of e^{φ} it is possible to get the embedding for harmonic functions with the constant $C = 4$. We suspect the constants e and 4 are the best possible for the analytic and harmonic embedding respectively, though this is still an open question. However, it is known that 4 is the best constant in the dyadic (martingale) Carleson Embedding Theorem.

The second example will appear in the proof of the Corona Theorem that we will see later.

Lemma 1.3. *Suppose that $\varphi \in H^\infty(\mathbb{D})$. Then for all $f \in H^2(\mathbb{D})$ we have*

$$\int_{\mathbb{D}} |f(z)|^2 |\varphi'(z)|^2 (1-|z|^2) dA(z) \lesssim \|\varphi\|_\infty^2 \|f\|_{H^2(\mathbb{D})}^2 .$$

Proof. The proof uses the alternate norm for $H^2(\mathbb{D})$. We have

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^2 |\varphi'(z)|^2 (1-|z|^2) dA(z) &= \int_{\mathbb{D}} |(f\varphi)'(z) - f'(z)\varphi(z)|^2 (1-|z|^2) dA(z) \\
&\leq \int_{\mathbb{D}} (|(f\varphi)'(z)|^2 + |f'(z)\varphi(z)|^2)(1-|z|^2) dA(z) \\
&\lesssim \|\varphi f\|_{H^2(\mathbb{D})}^2 + \|\varphi\|_\infty^2 \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2) dA(z) \\
&\lesssim \|\varphi\|_\infty^2 \|f\|_{H^2(\mathbb{D})}^2 .
\end{aligned}$$

□

A similar lemma can of course be proved by replacing $(1-|z|^2)$ with $\log \frac{1}{|z|}$ (just repeat the argument given).

Lemma 1.3 is a special case of a more general class of functions that generate Carleson measures for $H^2(\mathbb{D})$. Recall that a function $\varphi \in BMO(\mathbb{T})$ if

$$\|\varphi\|_{BMO}^2 = \sup_{z \in \mathbb{D}} |\varphi|^2(z) - |\varphi(z)|^2 < \infty,$$

where $\varphi(z)$ denotes the harmonic extension of φ to $\mathbb{D}$, and $|\varphi|^2(z)$ denotes the harmonic extension of $|\varphi(\xi)|^2$. This is typically called the Garsia norm of the function and is one of many useful norms on this space. Note that the expression on the right hand side of the definition of BMO is always non-negative since we are integrating against a probability measure and a simple application of Cauchy-Schwarz. One can also see that $L^\infty(\mathbb{T})$ is a subset of BMO.

Exercise 1.4. *Show that $L^\infty(\mathbb{T})$ is a subset of $BMO(\mathbb{T})$.*

Note that we have the following identity holding

$$\int_{\mathbb{T}} |\varphi(\xi) - \varphi(z)|^2 P_z(\xi) dm(\xi) = |\varphi|^2(z) - |\varphi(z)|^2 . \tag{1.1}$$

If we apply the conformally invariant version of Green's Theorem to the left hand side of (1.1) then we obtain

$$|\varphi|^2(z) - |\varphi(z)|^2 = \int_{\mathbb{D}} |\nabla \varphi(w)|^2 \log \left| \frac{1-\overline{z}w}{w-z} \right| dA(w) \tag{1.2}$$

Exercise 1.5. *Derive (1.2) from Green's formula for harmonic functions. Hint: First consider the case when $z = 0$ and then make a suitable change of variables.*

Using the relationship $\log \frac{1}{t} \approx 1 - t$ (which arose when we studied the equivalent norm on $H^2(\mathbb{D})$) and the identity

$$1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} = \left| \frac{z-w}{1-\overline{z}w} \right|^2$$

we see that

$$\int_{\mathbb{D}} |\nabla\varphi(w)|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w) \leq |\varphi|^2(z) - |\varphi(z)|^2 \leq C \int_{\mathbb{D}} |\nabla\varphi(w)|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w).$$

These computations then prove the following

Lemma 1.6. *Suppose that $\varphi \in BMO(\mathbb{T})$ then*

$$|\nabla\varphi(w)|^2 (1-|w|^2) dA(w)$$

is a $H^2(\mathbb{D})$ Carleson measure with Carleson measure norm controlled by a constant times $\|\varphi\|_{BMO}$.

The interested reader can see more about this proof in [1].

2. The Multiplier Algebra for $H^2(\mathbb{D})$

We now turn to studying the multiplier algebra for $H^2(\mathbb{D})$. First, recall that the multiplier algebra of a space $H^2(\mathbb{D})$ is the collection of functions such that

$$\mathcal{M}_{H^2(\mathbb{D})} := \{\varphi \in \mathrm{Hol}(\mathbb{D}) : \|\varphi f\|_{H^2(\mathbb{D})} \leq C \|f\|_{H^2(\mathbb{D})} \quad \forall f \in H^2(\mathbb{D})\}$$

We can place a norm on this space by defining

$$\|\varphi\|_{\mathcal{M}_{H^2}} := \inf\{C : \|\varphi f\|_{H^2(\mathbb{D})} \leq C \|f\|_{H^2(\mathbb{D})} \quad \forall f \in H^2(\mathbb{D})\}.$$

Exercise 2.1. *Show that with this definition that $\mathcal{M}_{H^2(\mathbb{D})}$ is a Banach algebra.*

Exercise 2.2. *Show that $H^\infty(\mathbb{D})$ is a Banach algebra.*

This is a very abstract way for us to study the space $H^\infty(\mathbb{D})$. Since, as we will see now these multiplier norm and the sup norm are in fact equal.

Proposition 2.3.

$$H^\infty(\mathbb{D}) = \mathcal{M}_{H^2(\mathbb{D})}$$

with

$$\|\varphi\|_{H^\infty(\mathbb{D})} = \|\varphi\|_{\mathcal{M}_{H^2}}.$$

Proof. First, observe that if $\varphi \in H^\infty(\mathbb{D})$ then we have for any $f \in H^2(\mathbb{D})$ that

$$\|\varphi f\|_{H^2(\mathbb{D})} \leq \|\varphi\|_{H^\infty(\mathbb{D})} \|f\|_{H^2(\mathbb{D})}.$$

This then gives that

$$\|\varphi\|_{H^\infty(\mathbb{D})} \geq \|\varphi\|_{\mathcal{M}_{H^2}}.$$

To prove the other inequality, note that $1 \in H^2(\mathbb{D})$ and so

$$\|\varphi\|_{H^2(\mathbb{D})} \leq \|\varphi\|_{\mathcal{M}_{H^2}}.$$

Using this we see that

$$\left\|\varphi^2\right\|_{H^2(\mathbb{D})} = \|\varphi\cdot\varphi\|_{H^2(\mathbb{D})} \leq \|\varphi\|_{H^2(\mathbb{D})} \|\varphi\|_{\mathcal{M}_{H^2}} \leq \|\varphi\|^2_{\mathcal{M}_{H^2}}.$$

Induction then gives that for any $n \in \mathbb{N}$ that

$$\|\varphi^n\|_{H^2(\mathbb{D})} \leq \|\varphi\|^n_{\mathcal{M}_{H^2}}.$$

So, we have that

$$\|\varphi^n\|^{\frac{1}{n}}_{H^2(\mathbb{D})} \leq \|\varphi\|_{\mathcal{M}_{H^2}}.$$

But, letting $n \to \infty$ we see that

$$\|\varphi\|_{H^\infty(\mathbb{D})} \leq \|\varphi\|_{\mathcal{M}_{H^2}}.$$

Here is an alternate way to see the last inequality. Let $\varphi \in \mathcal{M}_{H^2}$ and let M_φ be the operator of multiplication by φ acting on $H^2(\mathbb{D})$. Let M_φ^* be the adjoint of the operator M_φ. We select $\zeta, z \in \mathbb{D}$ and compute

$$\begin{aligned} M_\varphi^* k_\zeta(z) &= \left\langle M_\varphi^* k_\zeta, k_z \right\rangle \\ &= \langle k_\zeta, M_\varphi k_z \rangle_{H^2(\mathbb{D})} \\ &= \langle k_\zeta, \varphi k_z \rangle_{H^2(\mathbb{D})} \\ &= \overline{\langle \varphi k_z, k_\zeta \rangle}_{H^2(\mathbb{D})} \\ &= \overline{\varphi(\zeta) k_z(\zeta)} \\ &= \overline{\varphi(\zeta)} k_\zeta(z). \end{aligned}$$

Thus, k_ζ is an eigenvector of M_φ^*, the adjoint of the multiplication operator, with eigenvalue $\overline{\varphi(\zeta)}$. Hence, $|\varphi(\zeta)| \leq \left\|M_\varphi^*\right\| = \|M_\varphi\|$. Taking the supremum over $\zeta \in \mathbb{D}$ gives the desired estimate. □

References

[1] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. ↑3

[2] N. K. Nikol′skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. Spectral function theory; With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller; Translated from the Russian by Jaak Peetre. ↑1