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Hardy spaces associated to the Schrödinger operator on strongly Lipschitz domains of $\mathbb{R}^d$

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Abstract Let $L = -\Delta + V$ be a Schrödinger operator and Ω be a strongly Lipschitz domain of $\mathbb{R}^d$, where Δ is the Laplacian on $\mathbb{R}^d$ and the potential V is a nonnegative polynomial on $\mathbb{R}^d$. In this paper, we investigate the Hardy spaces on Ω associated to the Schrödinger operator L.

Keywords Schrödinger operator · Strongly Lipschitz domain · Hardy spaces · Boundary condition

Mathematics Subject Classification (2000) 35J10 · 42B25 · 42B30 · 42B35

1 Introduction

In recent years, a quite complete theory of Hardy spaces on domains has been developed [1,5,6]. The Hardy spaces are either defined in terms of restrictions to Ω of $H^1(\mathbb{R}^d)$ functions (which are denoted by $H_r(\Omega)$) or in terms of those $H^1(\mathbb{R}^d)$ functions having support in $\overline{\Omega}$ (which are denoted by $H_z(\Omega)$). In [1], the authors proved that the Laplacian doesn't play a specific role in characterizing the Hardy spaces on domains. They also showed that we can replace the Laplacian by a second-order elliptic operator. It is natural to ask whether we can replace the Laplacian by other operators, such as Schrödinger operators. In this paper, we will prove that under proper conditions the maximal function of Schrödinger operator can characterize the Hardy spaces defined in [9] on Strongly Lipschitz domains.

A strongly Lipschitz domain is defined by a domain in $\mathbb{R}^d$ whose boundary is covered by a finite number of parts of Lipschitz graphs (up to rotations) and at most one of them being unbounded. A special Lipschitz domain is the domain above the graph of a Lipschitz function defined on $\mathbb{R}^{d-1}$. Therefore, the special Lipschitz domains consistent a special class

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of strongly Lipschitz domains. Some basic properties about the strongly Lipschitz domain can be found in [1].

Let Ω be a strongly Lipschitz domain, we denote by $W^{1,2}(\Omega)$ the usual Sobolev space on Ω equipped with the norm $(\|f\|_2^2 + \|\nabla f\|_2^2)^{1/2}$ and $W_0^{1,2}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. If W is a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$, then we denote by L the maximal-accretive operator on $L^2(\Omega)$ with the largest domain $\mathcal{D}(L) \subset W$ such that

$$\langle Lf, g\rangle = \int_\Omega \nabla f \cdot \nabla g + V f \bar{g}, \quad \forall f \in \mathcal{D}(L), \ \ g \in W,$$

where V is a nonnegative polynomial. We use $L = (\Omega, V, W)$ to denote any operator defined as above. In the definition of Hardy spaces on domains, the boundary conditions play an important role. We will say that L satisfies Dirichlet boundary condition if $W = W_0^{1,2}(\Omega)$.

In the sequel, we use $L = -\Delta + V$ to denote a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where $V(x) = \sum_{\beta \leq \alpha} a_\beta x^\beta$ is a nonnegative polynomial on $\mathbb{R}^d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$, $\mathbb{N} = \{0, 1, 2, \ldots\}$. These operators have been studied by various authors (cf. [13,21,25]).

Let $\{T_t^L\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $K_t^L(x, y)$ be their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq K_t^L(x, y) \leq \widetilde{K_t}(x, y) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1}$$

In [9–11], the authors studied Hardy spaces and BMO spaces associated to Schrödinger operator L on $\mathbb{R}^d$. In this paper, we will investigate Hardy spaces associated to L on strongly Lipschitz domains.

We first recall some basic properties of Hardy and BMO spaces associated to L on $\mathbb{R}^d$(cf. [9,11]). We say that $f \in \mathcal{S}'(\mathbb{R}^d)$ is in the space $H_L^p(\mathbb{R}^d)$ if it satisfies

$$\mathcal{M} f(x) = \sup_{t>0} \left| T_t^L f(x) \right|$$

is in $L^p(\mathbb{R}^d)$, where $0 < p \leq 1$. The quasi-norm in H_L^p is defined by

$$\|f\|_{H_L^p}^p = \|\mathcal{M} f\|_{L^p}^p .$$

We define an auxiliary function $\rho(x, V)$ by

$$\rho(x, V) = \left(\sum_{\beta \leq \alpha} \left| D^\beta V(x) \right|^{1/(|\beta|+2)} \right)^{-1}, \tag{2}$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$.

Since $V(x)$ is a nonnegative polynomial, there is a constant $c > 0$ such that $\rho(x, V) \leq c$ for every $x \in \mathbb{R}^d$. We set

$$\mathcal{B}_0 = \{x \in \mathbb{R}^d : \ 1 < \rho(x, V) \leq c\}$$

and

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : \ 2^{-n/2} < \rho(x, V) \leq 2^{-(n-1)/2}\}$$

for $n = 1, 2, 3 \ldots$. Then we have $\mathbb{R}^d = \cup_{n=0}^\infty \mathcal{B}_n$. Let $B(x, r)$ be a ball in $\mathbb{R}^d$ with the center at x and radius r, for $0 < p \leq 1$, $1 \leq q \leq \infty$, we say a function a is an $H_L^{p,q}$-atom for

space $H^p_L(\mathbb{R}^d)$ associate to a ball $B(x_0, r)$ if

(1) $supp\ a \subset B(x_0, r)$,

(2) $\|a\|_{L^q} \le |B(x_0, r)|^{\frac{1}{q}-\frac{1}{p}}$,

(3) if $r < \rho(x_0)$, then $\int a(x)x^\gamma dx = 0$ for all $|\gamma| \le \left[d\left(\frac{1}{p}-1\right)\right]$.

The atomic quasi-norm in $H^p_L(\mathbb{R}^d)$ is defined by

$$\|f\|_{L-\text{atom},q} = \inf\left\{\left(\sum |c_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all decompositions $f = \sum c_j a_j$, where a_j are $H^{p,q}_L$-atoms.

Atomic decomposition for $H^p_L(\mathbb{R}^d)$ is stated as follows (see Theorem 1.12 in [9]).

Proposition 1 *Assume that $0 < p \le 1$, then the norms $\|f\|_{H^p_L}$ and $\|f\|_{L-\text{atom},\infty}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^p_L} \le \|f\|_{L-\text{atom},\infty} \le C\|f\|_{H^p_L}.$$

We say that a locally integrable function f belongs to $BMO_L(\mathbb{R}^d)$ if $f^\sharp_V \in L^\infty(\mathbb{R}^d)$, where

$$f^\sharp_V(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y) - f(B, V)|\, dy$$

and

$$f(B, V) = \begin{cases} \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} f(y)\, dy, & \text{if } r < \rho(x), \\ 0, & \text{if } r \ge \rho(x). \end{cases}$$

We set $\|f\|_{BMO_L} = \|f^\sharp_V\|_{L^\infty}$. It is clear that $L^\infty(\mathbb{R}^d) \subset BMO_L(\mathbb{R}^d) \subset BMO(\mathbb{R}^d)$ and $\|\varphi\|_{BMO} \le 2\|\varphi\|_{BMO_L}$. Also $\|\varphi\|_{BMO_L} = 0$ if and only if $\varphi(x) = 0$ a.e. $x \in \mathbb{R}^d$. The $BMO_L(\mathbb{R}^d)$ defined above is the dual space of $H^1_L(\mathbb{R}^d)$ and it has a Carleson measure characterization as classical BMO space (cf. [10]).

Let $\{P^L_t\}_{t>0}$ be the semigroup generated by $-\sqrt{L}$ and $P^L_t(x, y)$ be their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \le P^L_t(x, y) \le \widetilde{P}_t(x, y) = \frac{C_d\, t}{(t^2 + |x-y|^2)^{(d+1)/2}}. \tag{3}$$

Definition 1 Let $\frac{d}{d+1} < p \le 1$, we say f belongs to $H^p_{L,r}(\Omega)$ if it is the restriction to Ω of a function $F \in H^p_L(\mathbb{R}^d)$ and the quasi-norm is defined by

$$\|f\|_{H^p_{L,r}(\Omega)} = \inf\|F\|_{H^p_L(\mathbb{R}^d)},$$

where the infimum is taken over all the functions $F \in H^p_L(\mathbb{R}^d)$ such that $F|_\Omega = f$.

For $f \in L^1_{loc}(\Omega)$ and $|y|^{-d-1}f(y) \in L^1(\Omega)$, we define, for all $x \in \Omega$,

$$f^*_L(x) = \sup_{y\in\Omega, t>0, |x-y|<t}\left|P^L_t f(y)\right|.$$

We will say $f \in H^p_{\max,L}(\Omega)$ if $f^*_L \in L^p(\Omega)$ and define

$$\|f\|_{H^p_{\max,L}(\Omega)} = \left\|f^*_L\right\|_{L^p(\Omega)},$$

where $\frac{d}{d+1} < p \leq 1$.

In the sequel, we introduce area integral function associated to L. For $x \in \Omega$, set

$$S_L f(x) = \left(\int\limits_{\Gamma(x)} \left| t\overline{\nabla} P^L_t f(y) \right|^2 \frac{dydt}{t^{d+1}} \right)^{1/2},$$

where $\overline{\nabla} u = (\nabla u, \partial_t u)$, $|\overline{\nabla} u|^2 = |\nabla u|^2 + |\partial_t u|^2$ and $\Gamma(x)$ is the cone defined by $\Gamma(x) = \{(y,t) \in \Omega \times (0,\infty) : |x-y| < t\}$. We also define an area integral function restricted to time derivative as

$$s_L f(x) = \left(\int\limits_{\Gamma(x)} \left| t\partial_t P^L_t f(y) \right|^2 \frac{dydt}{t^{d+1}} \right)^{1/2}.$$

It is obvious that $s_L f \leq S_L f$ pointwise.

We define the Campanato space associated to Schrödinger operator as

Definition 2 Let $0 \leq \alpha < 1$, a locally integrable function g on $\mathbb{R}^d$ belongs to $\Lambda^L_\alpha(\mathbb{R}^d)$ if and only if $\|g\|_{\Lambda^L_\alpha(\mathbb{R}^d)} < \infty$, where

$$\|g\|_{\Lambda^L_\alpha(\mathbb{R}^d)} = \sup_{B \subset \mathbb{R}^d} \left\{ |B|^{-\frac{\alpha}{d}} \left(\int\limits_B |g - g(B,V)|^2 \frac{dx}{|B|} \right)^{1/2} \right\}.$$

Let $D^L_t = t\partial_t P^L_t$, then the following condition is very important for our proof.

Definition 3 We say $f \in \left(\Lambda^L_\alpha(\mathbb{R}^d)\right)^*$ vanishes at infinity in a generalized sense, if it satisfies

$$\lim_{A\to\infty} \int\limits_A^\infty \left\langle f, \left(D^L_t\right)^2 g \right\rangle \frac{dt}{t} = 0 \tag{4}$$

for all $g \in \Lambda^L_\alpha(\mathbb{R}^d)$.

The main result of this paper is the following theorem.

Theorem 1 *Let $\frac{d}{d+1} < p \leq 1$ and f be a locally integrable function on Ω that vanishes at infinity in a generalized sense, then if L satisfies the Dirichlet boundary condition on Ω and Ω^c is unbounded, one has*

$$\|f\|_{H^p_{L,r}(\Omega)} \leq C \|s_L f\|_{L^p(\Omega)} \leq C \|S_L f\|_{L^p(\Omega)} \leq C \|f\|_{H^p_{\max,L}(\Omega)} \leq C \|f\|_{H^p_{L,r}(\Omega)}.$$

The paper is organized as follows. In Sect. 2, we will introduce the dual and predual spaces of $H^p_L(\mathbb{R}^d)$. In Sect. 3, we give the proof of Theorem 1.

Throughout the article, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{B_1}{B_2} \leq C$.

2 Preliminaries

By the subordination formula and the estimates of the heat kernel (cf. [12]), we can get the following estimates of the Poisson kernel $P_t^L(x, y)$ on $\mathbb{R}^d$.

Proposition 2 (a) *For every $N > 0$, there exist constants $C_N > 0$ and $A > 0$ such that*

$$0 \le P_t^L(x, y) \le C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{5}$$

(b) *Let $0 < \delta < 1$ and $|h| < t$, then for any $N > 0$, there exist constants $A > 0$, $C_N > 0$, such that*

$$|P_t^L(x+h, y) - P_t^L(x, y)| \le C_N \left(\frac{|h|}{t}\right)^{\delta} \frac{t}{(t^2+A|x-y|^2)^{(d+1)/2}} \left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}. \tag{6}$$

Remark 1 We can replace $|h| < t$ by $|h| < \frac{|x-y|}{2}$ in Proposition 2.

Let $D_t^L(x, y)$ be the kernel of the operator $D_t^L = t\partial_t P_t^L$, then we have

Proposition 3 *There exists a constant $C > 0$, such that for every $N > 0$ and $0 < \delta < 1$, there is a constant $C_N > 0$, so that*

(a) $\left|D_t^L(x, y)\right| \le C_N \frac{t}{(t^2 + C|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}$;

(b) $|D_t^L(x+h, y) - D_t^L(x, y)| \le C_N \left(\frac{|h|}{t}\right)^{\delta} \frac{t}{(t^2+C|x-y|^2)^{(d+1)/2}} \left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}$,
for all $|h| \le t$;

(c) $\left|\int_{\mathbb{R}^d} D_t^L(x, y)dy\right| \le C_N \frac{t/\rho(x)}{(1 + t/\rho(x))^N}$.

We have the following proposition about $\rho(x)$ (cf. [21] Lemma 1.4).

Proposition 4 *There exist $C, k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}.$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

We can prove the following result similarly as in [16].

Proposition 5 *Let $0 < p \le 1$, then $\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)$ is the dual space of $H^p_L(\mathbb{R}^d)$.*

Definition 4 We will say a function $f \in \Lambda^L_\alpha(\mathbb{R}^d)$ is in $\lambda^L_\alpha(\mathbb{R}^d)$, if it satisfies $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0} \sup_{B\subset\mathbb{R}^d \ r_B<r} \frac{1}{|B|^{\alpha/d+1/2}} \left(\int_B |f - f(B,V)|^2 \frac{dxdt}{t}\right)^{1/2};$$

$$\gamma_2(f) = \lim_{r\to \infty} \sup_{B\subset\mathbb{R}^d \ r_B\geq r} \frac{1}{|B|^{\alpha/d+1/2}} \left(\int_B |f - f(B,V)|^2 \frac{dxdt}{t}\right)^{1/2};$$

$$\gamma_3(f) = \lim_{r\to \infty} \sup_{B\subset B(0,r)^c} \frac{1}{|B|^{\alpha/d+1/2}} \left(\int_B |f - f(B,V)|^2 \frac{dxdt}{t}\right)^{1/2}.$$

The dual of $\lambda^L_{d(1/p-1)}(\mathbb{R}^d)$ is $B^p_L(\mathbb{R}^d)$, where $B^p_L(\mathbb{R}^d)$ is the Banach complete space of $H^p_L(\mathbb{R}^d)$ (cf. [17]). In fact, we have the following theorem.

Theorem 2 *Let $\frac{d}{d+1} < p \leq 1$ and $\alpha = d(1/p-1)$, then*

(a) *Suppose $f \in H^p_L(\mathbb{R}^d)$. Then $\mathcal{L}_f = \int_{\mathbb{R}^d} f(x)g(x)dx$ defined initially for $g \in L^2_{loc}(\mathbb{R}^d)$ can be extended to a bounded linear functional on $\lambda^L_\alpha(\mathbb{R}^d)$ and satisfies*

$$\left\|\mathcal{L}_f\right\| \leq C \ \|f\|_{H^p_L(\mathbb{R}^d)}.$$

(b) *Conversely, every bounded linear functional $\mathcal{L}$ on $\lambda^L_\alpha(\mathbb{R}^d)$ can be realized as $\mathcal{L} = \mathcal{L}_f$ with $f \in H^p_L(\mathbb{R}^d)$ and*

$$\left\|f\right\|_{H^p_L(\mathbb{R}^d)} \leq C \left\|\mathcal{L}\right\|.$$

Remark 2 As we can only prove $\lambda^L_{d(1/p-1)}(\mathbb{R}^d)$ is the predual space of Hardy space $H^p_L(\mathbb{R}^d)$ for $\frac{d}{d+1} < p \leq 1$, our main result of this paper can only be obtained for $\frac{d}{d+1} < p \leq 1$. The main difficulty to extend Theorem 1 to $0 < p \leq \frac{d}{d+1}$ is that we can't get the Carleson measure characterization for $\Lambda^L_{d(1/p-1)}$.

3 The proofs of the main result

In this section, we prove the main result of this paper. We will divide it into several steps and our proof is motivated by [1].

3.1 From Hardy spaces to maximal Hardy spaces

Lemma 1 *Let $\frac{d}{d+1} < p \leq 1$ and f be a locally integrable function on Ω, then under DBC, one has*

$$\|f\|_{H^p_{\max,L}(\Omega)} \leq C \|f\|_{H^p_{L,r}(\Omega)}.$$

Proof When $f \in H^p_{L,r}(\Omega)$, it has an extension $\tilde{f} \in H^p_L(\mathbb{R}^d)$ with comparable norm. Then by Proposition 1, $\tilde{f}$ can be decomposed into $H^{p,\infty}_L$-atoms. Let $a(x)$ be a $H^{p,\infty}_L$-atom, then it is sufficient to prove

$$\|\chi_\Omega a\|_{H^p_{\max,L}(\Omega)} \leq C, \tag{7}$$

where C is a positive constant and independent of a. Assume $supp\, a \subset B(y_0, r)$.

If $B \subset \Omega$, then

$$P_t^L a(x) = \int_B P_t^L(x, y) a(y) dy.$$

Writing

$$\int_\Omega \left|a_L^*(x)\right|^p dx = \int_{B^* \cap \Omega} \left|a_L^*(x)\right|^p dx + \int_{\Omega \setminus B^*} \left|a_L^*(x)\right|^p dx = I_1 + I_2,$$

where $B^* = B(y_0, 8r)$.

For I_1, since $a_L^*(x) \leq C M(a)(x)$, where M is the Hardy–Littlewood maximal function, We have

$$I_1 \leq C \left|B^*\right|^{1-\frac{p}{2}} \left(\int_{\mathbb{R}^d} |M(a)(x)|^2 dx \right)^{p/2} \leq C |B|^{1-\frac{p}{2}} |B|^{\frac{p}{2}-1} = C. \tag{8}$$

For I_2, We will first prove

$$a_L^*(x) \leq C |B|^{\frac{1}{d}+1-\frac{1}{p}} |x - y_0|^{-d-1}. \tag{9}$$

For any $x \in \Omega \bigcap B(y_0, 2r)^c$, if $|x - y| < t$ and $|y_0 - z| < r$, we have

$$\begin{aligned} t + |y - z| &\geq t + |y - y_0| - |y_0 - z| \geq t + |x - y_0| - |x - y| - |y_0 - z| \\ &\geq |x - y_0| - r \geq \frac{|x - y_0|}{2}. \end{aligned}$$

In the sequel, we will consider two cases.

Case 1, when $r < \rho(y_0)$, $a(x)$ satisfies moment condition. Therefore, by Proposition 2 and Remark 1, we get

$$\begin{aligned} \left|P_t^L a(y)\right| &\leq \int_B \left|P_t^L(y, z) - P_t^L(y, y_0)\right| |a(z)| dz \\ &\leq C \int_B \frac{r}{(t^2 + A|y - z|^2)^{(d+1)/2}} |a(z)| dz \\ &\leq C |B|^{\frac{1}{d}+1-\frac{1}{p}} |x - y_0|^{-d-1}. \end{aligned} \tag{10}$$

Case 2, when $r \geq \rho(y_0)$, by Proposition 4, we know $\rho(z) \leq C r$ for any $z \in B(y_0, r)$. Therefore

$$\begin{aligned} \left| \int_B P_t^L(y, z) a(z) dz \right| &\leq C \int_B \frac{t}{\left(t^2 + A|y - z|^2\right)^{(d+1)/2}} \left(\frac{\rho(z)}{t}\right) |a(z)| dz \\ &\leq C \int_B \frac{r}{(t + |y - z|)^{d+1}} |a(z)| dz \\ &\leq C |B|^{\frac{1}{d}+1-\frac{1}{p}} |x - y_0|^{-d-1}. \end{aligned} \tag{11}$$

Taking supremum in $\{(y,t) \in \Omega \times (0,\infty) : |x-y| < t\}$ of (10) and (11), we get (9). Then,

$$\begin{aligned} I_2 &= \int\limits_{\Omega \setminus B^*} \left|a_L^*(x)\right|^p dx \\ &\le C\,|B|^{\frac{p}{d}+p-1} \int\limits_{(B^*)^c} \frac{1}{|x-y_0|^{(d+1)p}} dx \\ &\le C\,|B|^{\frac{p}{d}+p-1}\,|B|^{1-(1+\frac{1}{d})p} = C. \end{aligned}$$

If $B \bigcap \Omega = \phi$, there is nothing to prove. When $B \bigcap \partial\Omega \neq \phi$, we can choose $z' \in B \bigcap \partial\Omega$. By the Dirichlet boundary condition, we know $P_t^L(y,z') = 0$. When $|z-z'| < \frac{|y-z|}{2}$, by Remark 1, we have

$$\begin{aligned} \left| \int\limits_{B\cap\Omega} P_t^L(y,z)a(z)dz \right| &\le \int\limits_{B\cap\Omega} \left| P_t^L(y,z) - P_t^L(y,z') \right| |a(z)|dz \\ &\le C \int\limits_{B\cap\Omega} \frac{t}{\left(t^2 + A|y-z|^2\right)^{(d+1)/2}} \left(\frac{|z-z'|}{t}\right) |a(z)|\,dz \\ &\le C \int\limits_{B\cap\Omega} \frac{r}{(t+|y-z|)^{d+1}} |a(z)|\,dz \\ &\le C\,|B|^{\frac{1}{d}+1-\frac{1}{p}}\,|x-y_0|^{-d-1}. \end{aligned}$$

When $|z-z'| \ge \frac{|y-z|}{2}$, we have $|y-z| \le 4r$. Therefore, for $z \in B(y_0,r)$ and $|x-y| < t$, we have

$$t > |x-y| \ge |x-z| - |y-z| \ge 7r - 4r \ge \frac{3}{2}|z-z'|.$$

By Proposition 2 (b) and $|z-z'| < t$,

$$\begin{aligned} \left| \int\limits_{B\cap\Omega} P_t^L(y,z)a(z)dz \right| &= \left| \int\limits_{B\cap\Omega} \left(P_t^L(y,z) - P_t^L(z') \right) a(z)dz \right| \\ &\le \int\limits_{B\cap\Omega} \left| P_t^L(y,z) - P_t^L(y,z') \right| |a(z)|dz \\ &\le C \int\limits_{B\cap\Omega} \frac{t}{\left(t^2 + A|y-z|^2\right)^{(d+1)/2}} \left(\frac{|z-z'|}{t}\right) |a(z)|\,dz \\ &\le C \int\limits_{B\cap\Omega} \frac{r}{(t+|y-z|)^{d+1}} |a(z)|\,dz \\ &\le C\,|B|^{\frac{1}{d}+1-\frac{1}{p}}\,|x-y_0|^{-d-1}. \end{aligned}$$

Then, same as the proof of the case $B \subset \Omega$, we can prove $I_2 \le C$.
This proves Lemma 1. □

Remark 3 If we define the maximal function by heat kernel, then we can prove same result as Lemma 1 for $0 < p \le 1$.

3.2 From maximal functions to area integral functions

We first give the Caccioppoli inequality associated to L, which is very important for our proof (cf. [18]).

Let

$$Q\left((x_0,t_0),r\right)=\{(y,t):\max\{|y-x_0|,|t-t_0|\}<r\},$$

then, we have

Lemma 2 *Given $f\in L^2(\Omega)$, the solution $u(x,t)=e^{-t\sqrt{L}}f(x)$ of the solution of the equation $-\Delta u+\partial_t^2u+Vu=0$ in $\Omega\times\mathbb{R}^+$ satisfies*

$$\int\limits_{Q((x_0,t_0),r)\cap(\Omega\times\mathbb{R}^+)}\left|\overline{\nabla}u\right|^2+V(x)\left|u(x,t)\right|^2dxdt\leq\frac{C}{r^2}\int\limits_{Q((x_0,t_0),2r)\cap(\Omega\times\mathbb{R}^+)}\left|u(x,t)\right|^2dxdt.$$

Proof Let $\eta\in C_0^\infty\left(Q((x_0,t_0),2r)\right)$ that satisfies $0\leq\eta\leq1$ and $\eta(y,t)=1$ for $(y,t)\in Q((x_0,t_0),r)$, where $x_0\in\Omega$. Moreover, we assume $\left|\overline{\nabla}\eta\right|\leq\frac{C}{r}$.

In the sequel, we use $Q(2r)$ to denote $Q((x_0,t_0),2r)\bigcap(\Omega\times\mathbb{R}^+)$.

By the Dirichlet data on the lateral boundary $\partial\Omega\times(0,+\infty)$ and the supported condition of η, we get (cf. [2])

$$\begin{aligned}
0&=\int\limits_{Q(2r)}\overline{\nabla}u(x,t)\cdot\overline{\nabla}\left(u\eta^2\right)(x,t)+V(x)u\left(u\eta^2\right)(x,t)dx\,dt\\
&=\int\limits_{Q(2r)}\overline{\nabla}u(x,t)\cdot\left(\eta^2\overline{\nabla}u(x,t)+2\eta u\overline{\nabla}\eta(x,t)\right)dx\,dt\\
&\quad+\int\limits_{Q(2r)}V(x)\left|u^2\eta^2\right|(x,t)dx\,dt\\
&=\int\limits_{Q(2r)}\eta(x,t)\overline{\nabla}u(x,t)\cdot\left(\eta\overline{\nabla}u(x,t)+2u\overline{\nabla}\eta(x,t)\right)dx\,dt\\
&\quad+\int\limits_{Q(2r)}V(x)\left|u^2\eta^2\right|(x,t)dx\,dt\\
&=\int\limits_{Q(2r)}\left(\overline{\nabla}(u\eta)(x,t)-u\overline{\nabla}\eta(x,t)\right)\cdot\left(\overline{\nabla}(u\eta)(x,t)-u\overline{\nabla}\eta(x,t)+2u\overline{\nabla}\eta(x,t)\right)dx\,dt\\
&\quad+\int\limits_{Q(2r)}V(x)\left|u^2\eta^2\right|(x,t)dx\,dt\\
&=\int\limits_{Q(2r)}\left|\overline{\nabla}(u\eta)(x,t)\right|^2dx\,dt-\int\limits_{Q(2r)}\left|u(x,t)\right|^2\left|\overline{\nabla}\eta(x,t)\right|^2dx\,dt\\
&\quad+\int\limits_{Q(2r)}V(x)\left|u^2\eta^2\right|(x,t)dx\,dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{Q(r)} \left|\overline{\nabla} u(x,t)\right|^2 + V(x)\,|u(x,t)|^2\,dx\,dt \\
&\quad\le \int_{Q(2r)} \left|\overline{\nabla}(u\eta)(x,t)\right|^2 + V(x)\left|u^2\eta^2\right|(x,t)dx\,dt \\
&\quad= \int_{Q(2r)} |u(x,t)|^2 \left|\overline{\nabla}\eta(x,t)\right|^2 dx\,dt \le \frac{C}{r^2}\int_{Q(2r)} |u(x,t)|^2\,dx\,dt.
\end{aligned}$$

This proves Lemma 2. □

Let

$$S_{L,\alpha} f(x) = \left(\int_{\Gamma_\alpha(x)} \left| t\overline{\nabla} P_t^L f(y)\right|^2 \frac{dydt}{t^{d+1}} \right)^{1/2}$$

and

$$S_{L,\alpha}^{\epsilon,R} f(x) = \left(\int_{\Gamma_\alpha^{\epsilon,R}(x)} \left| t\overline{\nabla} P_t^L f(y)\right|^2 \frac{dydt}{t^{d+1}} \right)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y,t) \in \Omega \times \mathbb{R}^+ : |y-x| < \alpha t\}$ and $\Gamma_\alpha^{\epsilon,R}(x) = \{(y,t) \in \Omega \times (\epsilon, R) : |y-x| < \alpha t\}$. By Lemma 2, we can prove the following Lemma.

Lemma 3 *Assume $\alpha < 1$, then, for $f \in L^2(\Omega)$, we have*

$$S_\alpha^{\epsilon,R} f(x) \le C_\alpha\,(1 + |\ln(R/\epsilon)|)^{1/2}\, f_L^*(x)$$

for some $C_\alpha > 0$ that is independent of f.

Proof For $(z,\tau) \in \Gamma_\alpha^{\epsilon,R}(x)$, let $E_{(z,\tau)} = B((z,\tau),r) \bigcap (\Omega \times (0,\infty))$, where $r = \delta\tau$ and $B((z,\tau),r) = \{(y,t) : \sup\{|y-z|, |t-\tau|\} < r\}$, δ is a small positive constant. By Besicovitch covering Lemma, we can choose a subcollection $E_j = E_{(z_j,\tau_j)}$ covering $\Gamma_\alpha^{\epsilon,R}(x)$ and having bounded overlap. For $(y,t) \in E_j$, we know $|y-z_j| < \delta\tau_j$ and $|t-\tau_j| < \delta\tau_j$, therefore, $t \sim \tau_j$. Let d_j be the distance from E_j to the bottom boundary $\Omega \times \{0\}$, then $d_j \sim \tau_j$. Therefore $t \sim d_j$. If $(y,t) \in E_j^* = B\left((z_j,\tau_j), 2r_j\right) \bigcap (\Omega \times (0,\infty))$, then we can choose δ to be small enough such that

$$|x-y| \le |x-z_j| + |z_j - y| \le \alpha\tau_j + 2\delta\tau_j < t.$$

i.e., $(y,t) \in \Gamma_1(x)$. Hence $|P_t^L f(y)| \le f_L^*(x)$. Then, by the bounded overlap of $\{E_j\}$ and Lemma 2,

$$\begin{aligned}
S_\alpha^{\epsilon,R} f(x)^2 &\le C\sum_j \int_{E_j^*} t^{1-d} \left|\overline{\nabla} P_t^L f(y)\right|^2 dydt \\
&\le C\sum_j d_j^{1-d} r_j^{-2} \left|E_j^*\right| f_L^*(x)^2.
\end{aligned}$$

If $R > 3\epsilon$, we have

$$\sum_j d_j^{1-d} r_j^{-2} \int_{E_j} dydt \le C \sum_j \int_{E_j} \frac{dy\,dt}{t^{d+1}} \le C \int_{\Gamma_\alpha^{\epsilon,R}} \frac{dy\,dt}{t^{d+1}} \le C \ln(R/\epsilon).$$

If $R \le 3\epsilon$, then

$$\sum_j d_j^{1-d} r_j^{-2} \left|E_j\right| \le C\epsilon^{-d-1} \int_{\Gamma_\alpha^{\epsilon,R}} dydt \le C \frac{R^{d+1} - \epsilon^{d+1}}{\epsilon^{d+1}} \le C.$$

Therefore

$$\sum_j d_j^{1-d} r_j^{-2} \left|E_j^*\right| \le C\left(1 + |\ln(R/\epsilon)|\right).$$

This completes the proof of Lemma 3. □

For $x \in \Omega$, let

$$\widetilde{S}_\alpha^{\epsilon,R} f(x) = \left(\int_1^2 \int_{\Gamma_{\alpha/a}^{a\epsilon,aR}(x)} t^{1-d} \left|\overline{\nabla} P_t^L f(y)\right|^2 dydtda \right)^{1/2}.$$

It is easy to see that, for any $x \in \Omega$,

$$S_{\alpha/2}^{2\epsilon,R} f(x) \le \widetilde{S}_\alpha^{\epsilon,R} f(x) \le S_{2\alpha}^{\epsilon,2R} f(x) \tag{12}$$

Now, we can prove good λ inequality as follows.

Lemma 4 *There exists $C > 0$ such that, for all $0 < \gamma < 1$, $\lambda > 0$, $0 < \epsilon < R < \infty$ and $f \in H^p_{\max,L} \bigcap L^2(\Omega)$, we have*

$$\left|\left\{x \in \Omega : \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda,\, f_L^*(x) \le \gamma\lambda\right\}\right| \le C\gamma^2 \left|\left\{x \in \Omega : \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda\right\}\right|.$$

Proof Let ϵ, R and λ be fixed. Then for $f \in H^p_{\max,L} \bigcap L^2(\Omega)$, define $O = \{x \in \Omega : \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda\}$. We assume that $O \ne \Omega$, otherwise, there is nothing to prove. Let $O = \bigcup_k Q_k$ be a Whitney decomposition of O (with respect to Ω) by dyadic cubes (of $\mathbb{R}^d$), so that, for all k, $2Q_k \subset O \subset \Omega$, but $4Q_k$ intersects $\Omega \bigcap O^c$. Since

$$\left\{x \in \Omega : \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda\right\} \subset \left\{x \in \Omega : \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda\right\},$$

it is sufficient to prove

$$\left|\left\{x \in Q_k : \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda,\, f_L^*(x) \le \gamma\lambda\right\}\right| \le C\gamma^2 \left|Q_k\right|. \tag{13}$$

Fixing k and let l be the side length of Q_k. If $x \in Q_k$, we have

$$\widetilde{S}_{1/20}^{\sup\{10l,\epsilon\},R} f(x) \le \lambda.$$

In fact, choose $x_k \in 4Q_k$ and $x_k \notin O$. If $|x-y| < \frac{t}{20}$ and $t \geq \sup\{10l, \epsilon\}$, then we have

$$|x_k - y| < |x-y| + |x-x_k| < \frac{t}{20} + 4l < \frac{t}{2}.$$

Therefore,

$$\widetilde{S}^{\sup\{10l,\epsilon\},R}_{1/20} f(x) \leq \widetilde{S}^{\sup\{10l,\epsilon\},R}_{1/2} f(x_k) \leq \lambda.$$

If $\epsilon \geq 10l$, then $\left\{x \in Q_k : \widetilde{S}^{\epsilon,R}_{1/20} f(x) > 2\lambda\right\} = \phi$. So (13) holds.

If $\epsilon < 10l$, then

$$\widetilde{S}^{\epsilon,R}_{1/20} f(x) \leq \widetilde{S}^{\epsilon,10l}_{1/20} f(x) + \widetilde{S}^{10l,R}_{1/20} f(x).$$

Therefore, we only need to prove

$$\left|\left\{x \in Q_k : \widetilde{S}^{\epsilon,10l}_{1/20} f(x) > \lambda,\, f^*_L(x) \leq \gamma\lambda\right\}\right| \leq C\gamma^2 |Q_k|. \tag{14}$$

We denote

$$g(x) = \widetilde{S}^{\epsilon,10l}_{1/20} f(x), \qquad \text{and} \quad F = \left\{x \in \Omega : f^*_L(x) \leq \gamma\lambda\right\}.$$

As $(x,t) \to u_t(x) = P^L_t f(x)$ is a continuous function, we know F is a closed subset of Ω. By

$$|\{x \in Q_k \cap F : g(x) > \lambda\}| \leq \frac{1}{\lambda^2} \int\limits_{Q_k \cap F} |g(x)|^2\, dx,$$

we can get (14) from

$$\int\limits_{Q_k \bigcap F} |g(x)|^2\, dx \leq C\gamma^2\lambda^2 |Q_k|. \tag{15}$$

If $5l \leq \epsilon$, then by Lemma 3,

$$\int\limits_{Q_k \bigcap F} |g(x)|^2\, dx \leq C \int\limits_{Q_k \bigcap F} f^*_L(x)^2 dx \leq C\gamma^2\lambda^2 |Q_k \cap F|.$$

If $\epsilon < 5l$, then

$$\int\limits_{Q_k \bigcap \Omega} |g(x)|^2\, dx \leq C \int\limits_1^2 \int\limits_{\xi_a} t \left|\overline{\nabla} P^L_t f(y)\right|^2 dydtda,$$

where $\xi_a = \{(y,t) \in \Omega \times (a\epsilon, 10al) : a\psi(y) < t\}$ and $\psi(y) = 20d(y, Q_k \bigcap F)$.

It is obviously that $\xi_a = \{(y, at) : (y,t) \in \xi_1\}$. Let $E = \{y : (y,t) \in \xi_1\}$, then E is an open subset of Ω. For any connected component G of E, we let $\mathcal{L}_a = \{(y,t) \in \xi_a : y \in G\}$. It is sufficient to prove that

$$\int\limits_1^2 \int\limits_{\mathcal{L}_a} t \left|\overline{\nabla} u_t(y)\right|^2 dydtda \leq c\lambda^2\gamma^2 |G|. \tag{16}$$

In fact, if we can prove (16), we sum over all the connected components of E, then we can get

$$\int_1^2 \int_{\xi_a} t \left|\overline{\nabla} u_t(y)\right|^2 dydtda \le c\lambda^2\gamma^2 |E|.$$

If $y \in E$, then there exists a point $(y,t) \in \xi_1$. Therefore, we can find $x \in Q_k \bigcap F$ such that $|x-y| < \frac{t}{20}$. As $t < 10l$, we have $|x-y| < \frac{l}{2}$, i.e., $E \subset 2Q_k$, which proves that (15) is true.

We fix a connected component G of E, consider $a \in (1,2)$ and note that $\mathcal{L}_a$ is connected and has Lipschitz boundary. By

$$-(-\Delta + V)\, u_t(x) + \partial_t^2 u_t(x) = 0$$

in the weak sense on $\Omega \times (0,\infty)$, we get

$$\left|\overline{\nabla} u_t(x)\right|^2 = \frac{1}{2}\left((\Delta - V(x))u_t^2(x) + \partial_t^2 u_t^2(x)\right).$$

Therefore

$$\left|\overline{\nabla} u_t(x)\right|^2 \le \frac{1}{2}\Delta u_t^2(x) + \frac{1}{2}\partial_t^2 u_t^2(x).$$

Then by Green's formula, we have

$$\begin{aligned}\int_{\mathcal{L}_a} t \left|\overline{\nabla} u_t(y)\right|^2 dydt &\le \frac{1}{2}\int_{\mathcal{L}_a} t\Delta u_t^2(y) + t\partial_t^2 u_t^2(y) dydt \\ &= \int_{\partial\mathcal{L}_a} t u_t(y)\overline{\nabla} u_t(y)\cdot N_a(y,t) d\sigma_a(y,t) \\ &\quad + \frac{1}{2}\int_{\partial\mathcal{L}_a} u_t^2(y) N_a(y,t)\cdot(0,\dots,0,1) d\sigma_a(y,t), \end{aligned} \tag{17}$$

where $N_a(y,t)$ is the unit normal vector outward $\mathcal{L}_a$ and $d\sigma_a(y,t)$ is the surface measure over $\partial\mathcal{L}_a$.

In the following, we prove that $y \in 2Q_k \subset \Omega$ and $(y,t) \in \xi_1$ for $(y,t) \in \overline{\mathcal{L}_a}$. In fact, by the definition of $\mathcal{L}_a$ and note that F is a closed subset of Ω, we know that there exists $x \in Q_k \bigcap F$ such that $|x-y| \le \frac{t}{20a}$. Since $t < 10la$, we have $|x-y| < \frac{l}{2}$, this proves that $y \in 2Q_k \subset \Omega$. By $|x-y| \le \frac{t}{20a} < t$, we get $(y,t) \in \xi_1$. Therefore, $\overline{\mathcal{L}_a}$ remains far from the boundary of $\Omega \times (0,\infty)$, then we can do not care about the boundary values of $u_t(y)$.

By (17),

$$\int_{\mathcal{L}_a} t \left|\overline{\nabla} u_t(y)\right|^2 dydt \le C\int_{\partial\mathcal{L}_a} t\,|u_t(y)|\left|\overline{\nabla} u_t(y)\right| d\sigma_a(y,t) + \int_{\partial\mathcal{L}_a} |u_t(y)|^2 d\sigma_a(y,t).$$

Since $|u_t(y)| \le \gamma\lambda$ on $\partial\mathcal{L}_a$, we have

$$\int_1^2 \int_{\partial\mathcal{L}_a} |u_t(y)|^2 d\sigma_a(y,t)da \le \gamma^2\lambda^2 \int_1^2 \int_{\partial\mathcal{L}_a} d\sigma_a(y,t)da.$$

We will prove that

$$\int_1^2 \int_{\partial \mathcal{L}_a} d\sigma_a(y,t)da \le C\,|G|. \tag{18}$$

It is easy to know

$$\int_1^2 \int_{\partial \mathcal{L}_a} d\sigma_a(y,t)da \le C \int_{\mathcal{G}} \frac{dz\,ds}{s},$$

where $\mathcal{G}$ is the union of the sets $\partial\mathcal{L}_a$ for $1 < a < 2$. Then

$$\mathcal{G} = \{(z,s) : z \in G \text{ and } \epsilon < s < 2\epsilon \text{ or } \psi(z) < s < 2\psi(z) \text{ or } 10l < s < 20l\}.$$

Therefore

$$\int_1^2 \int_{\partial \mathcal{L}_a} d\sigma_a(y,t)da \le C \int_{\mathcal{G}} \frac{dz\,ds}{s} \le C\,|G|.$$

It remains to prove

$$\int_1^2 \int_{\partial \mathcal{L}_a} t\,|u_t(y)|\left|\overline{\nabla} u_t(y)\right| d\sigma_a(y,t)da \le C\,\gamma^2\lambda^2\,|G|\,. \tag{19}$$

Let $\mathcal{G}$ be the same set as above, then

$$\int_1^2 \int_{\partial \mathcal{L}_a} t\,|u_t(y)|\left|\overline{\nabla} u_t(y)\right| d\sigma_a(y,t)da \le C\,\gamma\lambda \int_{\mathcal{G}} \left|\overline{\nabla} u_t(y)\right| dydt.$$

Choosing a covering of $\mathcal{G}$ with bounded overlap by balls $B_j = B\left((x_j,t_j), \frac{\epsilon t_j}{20}\right)$. Noting that $(x,t) \in B_j$ implies $t \sim t_j \sim r_j$, where r_j is the radius of B_j. Then by Hölder's inequality and Caccioppoli's inequality, we get

$$\begin{aligned}
\int_{\mathcal{G}} \left|\overline{\nabla} u_t(y)\right| dydt &\le c \sum_j \int_{B_j} \left|\overline{\nabla} u_t(y)\right| dydt \\
&\le c \sum_j \left|B_j\right|^{1/2} \left(\int_{B_j} \left|\overline{\nabla} u_t(y)\right|^2 dydt \right)^{1/2} \\
&\le c \sum_j \left|B_j\right|^{1/2} r_j^{-1} \left(\int_{2B_j} |u_t(y)|^2\, dydt \right)^{1/2} \\
&\le c\gamma\lambda \sum_j \left|B_j\right| r_j^{-1} \le c\gamma\lambda \int_{\widetilde{\mathcal{G}}} \frac{dzds}{s},
\end{aligned}$$

where $\widetilde{\mathcal{G}}$ is a set like $\mathcal{G}$ but slight enlarged: it is contained set of points (z, s) with $z \in G$ and $\epsilon/2 < s < 4\epsilon$ or $\psi(z)/2 < s < 4\psi(z)$ or $5l < s < 40l$.

This proves (19) holds and then Lemma 4 is proved. □

The proof of the following Lemma can be found in [7].

Lemma 5 *For $\alpha, \beta > 0$ and $0 < \epsilon < R < \infty$, $0 < p < \infty$, we have*

$$\left\| S_\alpha^{\epsilon,R} \right\|_{L^p(\Omega)} \sim \left\| S_\beta^{\epsilon,R} \right\|_{L^p(\Omega)},$$

where the implicit constants do not dependent on ϵ, R, f.

Now, we can prove

Lemma 6 *Let $\frac{d}{d+1} < p \le 1$ and f be a locally integrable function on Ω, then under DBC, one has*

$$\|S_L f\|_{L^p(\Omega)} \le C \left\| f_L^* \right\|_{L^p(\Omega)}.$$

Proof Let $f \in H^p_{\max,L} \bigcap L^2(\Omega)$. By Lemma 4,

$$\left\| \widetilde{S}_{1/20}^{\epsilon,R} f \right\|_{L^p(\Omega)} \le C\, \gamma^{-1} \left\| f_L^* \right\|_{L^p(\Omega)} + C\, \gamma^2 \left\| \widetilde{S}_{1/2}^{\epsilon,R} f \right\|_{L^p(\Omega)}. \tag{20}$$

Following from Lemma 5 and (12),

$$\left\| S_1^{\epsilon,R} f \right\|_{L^p(\Omega)} \le C \left\| S_{1/40}^{\epsilon,R} f \right\|_{L^p(\Omega)} \le C \left\| \widetilde{S}_{1/20}^{\epsilon,R} f \right\|_{L^p(\Omega)}.$$

Then by Lemma 3,

$$\begin{aligned}
\left\| \widetilde{S}_{1/2}^{\epsilon,R} f \right\|_{L^p(\Omega)} &\le C \left\| S_1^{\epsilon/2,2R} f \right\|_{L^p(\Omega)} \\
&\le C \left\| S_1^{\epsilon/2,\epsilon} f \right\|_{L^p(\Omega)} + \left\| S_1^{\epsilon,R} f \right\|_{L^p(\Omega)} + \left\| S_1^{R,2R} f \right\|_{L^p(\Omega)} \\
&\le C \left\| S_1^{\epsilon,R} f \right\|_{L^p(\Omega)} + C \left\| f_L^* \right\|_{L^p(\Omega)} \\
&\le C \left\| \widetilde{S}_{1/20}^{\epsilon,R} f \right\|_{L^p(\Omega)} + C \left\| f_L^* \right\|_{L^p(\Omega)}.
\end{aligned}$$

We can choose a proper γ in (20) to get Lemma 6.

In the following, we relax the assumption $f \in L^2(\Omega)$. By $f_L^* \in L^p(\Omega)$, we get $P_s^L f \in L^p(\Omega)$ for any $s > 0$. Moreover, for any $s > 0$ and $x \in \Omega$, we have

$$\left| P_s^L f(x) \right| \le f_L^*(y), \qquad y \in B(x, s) \cap \Omega.$$

Therefore, by Lemma 7,

$$P_s^L f(x) \le \frac{1}{|B(x,s) \cap \Omega|} \int\limits_{B(x,s)\cap\Omega} f_L^*(y)^p dy \le \frac{c}{s^d} \int\limits_{\Omega} f_L^*(y)^p dy.$$

This proves $P_s^L f \in L^\infty(\Omega)$ for any $s > 0$. Then, we can get $P_s^L f \in L^2(\Omega)$ for any $s > 0$. By the proof of above, we have

$$\|S_L f_s\|_{L^p(\Omega)} \le C \left\| (f_s)_L^* \right\|_{L^p(\Omega)},$$

where $f_s = P_s^L f$. Since

$$(f_s)_L^*(x) = \sup_{y\in\Omega, t>0, |x-y|<t} \left|P_t^L f_s(y)\right| = \sup_{y\in\Omega, t>0, |x-y|<t} \left|P_{t+s}^L f(y)\right|$$
$$\leq \sup_{y\in\Omega, t>0, |x-y|<t+s} \left|P_{t+s}^L f(y)\right| = f_L^*(x),$$

we have

$$\|S_L f_s\|_{L^p(\Omega)} \leq C \left\|f_L^*\right\|_{L^p(\Omega)}.$$

Noting that

$$\lim_{s\to 0} P_t^L f_s(y) = \lim_{s\to 0} P_s^L P_t^L f(y) = P_t^L f(y), \qquad \text{in } \mathcal{S}'(\Omega),$$

we get

$$\lim_{s\to 0} S_L f_s(x) = \lim_{s\to 0} \int_0^\infty \int_{y\in\Omega, |x-y|<t} t^{1-d} \left|\overline{\nabla} P_t^L f_s(y)\right|^2 dydt$$
$$= \int_{y\in\Omega, |x-y|<t} t^{1-d} \left|\overline{\nabla} P_t^L f(y)\right|^2 dydt.$$

Then, Lemma 6 follows from monotone theorem. □

3.3 From area integral functions to Hardy spaces

In order to prove the final part of Theorem 1, we need to use the tent spaces on Ω. As the strongly Lipschitz domain Ω can be seen as a homogeneous space, we will give some properties of tent spaces on the homogeneous space. The tent spaces on $\mathbb{R}^d$ were introduced and developed by Coifman et al. [7]. Recently, Russ has studied the tent spaces on spaces of homogeneous type in [20].

We give some notations. Let (X, d) be a non-empty metric space endowed with a σ-finite Borel measure μ. For all $x \in X$ and all $r > 0$, denote by $B(x, r)$ the open ball centered at x with radius r, and by $V(x, r)$ its measure. We call (X, d, μ) a space of homogeneous type if, for all $x \in X$ and all $r > 0$, $V(x, r) < \infty$ and there exists $C < 0$ such that, for all $x \in X$ and all $r > 0$,

$$V(x, 2r) \leq CV(x, r). \tag{21}$$

An easy consequence of (21) is that there exist $C, D > 0$ such that, for all $x \in X$, $r > 0$ and $\theta > 1$,

$$V(x, \theta r) \leq C\theta^D V(x, r). \tag{22}$$

Now we define the tent spaces on X. For any $\alpha > 0$ and any $x \in X$, denote by $\Gamma_\alpha(x)$ the cone of aperture α with vertex $x \in X$, defined by

$$\Gamma_\alpha(x) = \{(y, t) \in X \times (0, \infty) | \quad d(x, y) < \alpha t\}.$$

We use $\Gamma(x)$ to denote $\Gamma_1(X)$. For any measurable function f on $X\times(0,\infty)$ and any $x\in X$, define

$$Sf(x)=\left(\int\limits_{\Gamma(x)}\frac{|f(y,t)|^2}{V(x,t)}d\mu(y)\frac{dt}{t}\right)^{1/2},$$

and, for $0<p\le 1$, say that $f\in T_2^p(X)$ if

$$\|f\|_{T_2^p(X)}:=\|Sf\|_{L^p(X)}<\infty.$$

Let $T(B)$ denote the tent of a ball $B(x,r)\subset X$, i.e., $T(B)=\{(y,t)\in X\times(0,\infty):\ d(x,y)<r-t\}$. Then we can define atoms for T_2^p as

Definition 5 Let $0<p\le 1$. A measurable function a on $X\times(0,\infty)$ is said to be a $T_2^p(X)$ atom if there exists a ball $B\subset X$ such that a is supported in $T(B)$ and

$$\int\limits_{X\times(0,\infty)}|a(x,t)|^2\,d\mu(y)\frac{dt}{t}\le\frac{1}{V(B)^{2/p-1}}.$$

The main result of [20] is the following atomic decomposition for $T_2^p(X)$.

Proposition 6 *Let $0<p\le 1$. Then, there exists $C_p>0$ with the following property: for all $f\in T_2^p(X)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}\in l^p$ and a sequence of $T_2^p(X)$-atoms $(a_n)_{n\in\mathbb{N}}$ such that*

$$f=\sum_n\lambda_n a_n\quad\text{and}\quad\sum_n|\lambda_n|^p\le C_p^p\,\|f\|^p_{T_2^p(X)}.$$

Let

$$T_2^{p,\infty}=\left\{f(x,t):\ \text{measurable on}\ \ X\times(0,\infty)\ \text{and}\ \|f\|_{T_2^{p,\infty}}<\infty\right\},$$

where

$$\|f\|_{T_2^{p,\infty}}=\sup_{B\subset X}\frac{1}{|V(B)|^{1/p-1/2}}\left(\int\limits_{T(B)}|f(x,t)|^2\,\frac{d\mu(x)dt}{t}\right)^{1/2}.$$

We use $T_{2,c}^p$ to denote the set of all $f\in T_2^p$ with compact support in $X\times(0,\infty)$ and $\widetilde{T_2^p}=\{F(x,t)=\sum_i\lambda_i a_i(x,t):\ a_i(x,t)$ are T_2^p atoms and $\sum_i|\lambda_i|^p<\infty\}$. Let $\|F\|_{\widetilde{T_2^p}}=\inf\left\{\left(\sum_i|\lambda_i|^p\right)^{1/p}:\ F(x,t)=\sum_i\lambda_i a_i(x,t)\right\}$, then $\widetilde{T_2^p}$ is a Banach space. In fact, it is the completeness of T_2^p. In particular, we have $\widetilde{T_2^1}=T_2^1$.

Theorem 3 *Let $0<p\le 1$, then*

(1) *For any $g\in T_2^{p,\infty}$, let*

$$\mathcal{L}_g(f)=\int\limits_{X\times(0,\infty)}f(x,t)g(x,t)\frac{d\mu(x)dt}{t},$$

where $f\in T_{2,c}^p$. Then $\mathcal{L}_g$ can be extended to a continuous linear functional on T_2^p, moreover, we have

$$\left| \int\limits_{X\times(0,\infty)} f(x,t)g(x,t)\frac{d\mu(x)dt}{t} \right| \le C\,\|g\|_{T_2^{p,\infty}}\,\|f\|_{T_2^p}\,.$$

(2) *For* $\mathcal{L} \in \left(T_2^p\right)^*$, *there exists* $g \in T_2^{p,\infty}$ *such that for any* $f \in T_2^p$, *we have*

$$\mathcal{L}(f) = \int\limits_{X\times(0,\infty)} f(x,t)g(x,t)\frac{d\mu(x)dt}{t}$$

and $\|g\|_{T_2^{p,\infty}} \le C\|\mathcal{L}\|$.

Proof For any $f \in T_2^p$, we have atomic decomposition

$$f(x,t) = \sum_i \lambda_i a_i(x,t) \qquad \text{and} \quad \sum_i |\lambda_i|^p \le C\,\|f\|_{T_2^p}^p\,.$$

Then, for $g \in T_2^{p,\infty}$, we have

$$\begin{aligned}
\left|\mathcal{L}_g(f)\right| &= \left| \int\limits_{X\times(0,\infty)} f(x,t)g(x,t)\frac{d\mu(x)dt}{t} \right| \\
&\le \sum_i |\lambda_i| \int\limits_{X\times(0,\infty)} |a_i(x,t)|\,|g(x,t)|\,\frac{d\mu(x)dt}{t} \\
&\le \sum_i |\lambda_i| \left(\int\limits_{X\times(0,\infty)} |a_i(x,t)|^2\,\frac{d\mu(x)dt}{t} \right)^{1/2} \left(\int\limits_{T(B_j)} |g(x,t)|^2\,\frac{d\mu(x)dt}{t} \right)^{1/2} \\
&\le \sum_i |\lambda_i|\,\|g\|_{T_2^{p,\infty}} \le \left(\sum_i |\lambda_i|^p \right)^{1/p} \|g\|_{T_2^{p,\infty}}\,,
\end{aligned}$$

where B_j is a ball in X such that $T(B_j)$ is the support of a_j. This proves part (1) of Theorem 3.

Let K be any compact subset of $X \times (0,\infty)$, then for any $f \in L^2\left(K, \frac{d\mu(x)dt}{t}\right)$, we have $F = f\chi_K \in T_2^p$. In fact, by Hölder inequality and Fubini's theorem,

$$\begin{aligned}
\|F\|_{T_2^p} &= \left(\int\limits_X \left(\int\limits_{\Gamma(x)} \frac{|F(y,t)|^2}{V(x,t)}\,\frac{d\mu(y)dt}{t} \right)^{p/2} d\mu(x) \right)^{1/p} \\
&\le C_K \left(\int\limits_X \left(\int\limits_{\Gamma(x)} \frac{|F(y,t)|^2}{V(x,t)}\,\frac{d\mu(y)dt}{t} \right) d\mu(x) \right)^{1/2} \\
&= C_K \left(\int\limits_0^\infty \int\limits_X \frac{|F(y,t)|^2}{V(x,t)}\,\mu\left(\{x\in X:\ d(y,x) < t\}\right)\frac{d\mu(y)dt}{t} \right)^{1/2} \\
&\le C_K\,\|f\|_{L^2(K,\frac{d\mu(x)dt}{t})}.
\end{aligned}$$

Therefore $\mathcal{L}$ can be acted on $L^2\left(K, \frac{d\mu(x)dt}{t}\right)$. Then we get $g_K \in L^2\left(K, \frac{d\mu(x)dt}{t}\right)$ such that

$$\mathcal{L}(F) = \int\limits_{X\times(0,\infty)} F(x,t)g_K(x,t)\frac{d\mu(x)dt}{t}, \qquad \text{for } F \in L^2\left(K, \frac{d\mu(x)dt}{t}\right).$$

Let $\{K_n\}$ be an increasing sequence of compact subsets of $X \times (0, +\infty)$ such that

$$\bigcup_{n\geq 1} K_n = X \times (0, +\infty).$$

There exist $g_{K_n} \in L^2\left(K_n, \frac{d\mu(x)dt}{t}\right)$ with $g_{K_n}|_{K_{n-1}} = g_{K_{n-1}}$ and

$$\mathcal{L}(G) = \int\limits_{X\times(0,\infty)} G(x,t)g_{K_n}(x,t)\frac{d\mu(x)dt}{t}, \qquad \text{for } G \in L^2\left(K_n, \frac{d\mu(x)dt}{t}\right), \quad n = 1, 2, \ldots$$

Then, there exists $g \in L^2_{loc}(X \times (0, +\infty))$ such that $g|_{K_n} = g_{K_n}$.

For any compact set $K \subset T(B)$ and $supp\ h \subset K$ with $\|h\|_{L^2(K)} \leq 1$, $a(x,t) = |V(B)|^{1/2-1/p}h(x,t)$ is a constant times a T_2^p atom that supported in $T(B)$. Assume $K \subset K_n$ and let $G = a$, then

$$\left(\int\limits_K |g(x,t)|^2 \frac{d\mu(x)dt}{t}\right)^{1/2} = \sup_{\|h\|_{L^2(K)}\leq 1} \int\limits_K g(x,t)h(x,t)\frac{d\mu(x)dt}{t} \leq C\|\mathcal{L}\||V(B)|^{1/p-1/2}.$$

By an argument similar to the proof of Proposition 4.2 in [3], for any $g \in L^2\left(T(B), \frac{d\mu(x)dt}{dt}\right)$, there exist $\{g_n\}$ with g_n is bounded and has compact support in $T(B)$ such that $g_n \to g$ in $L^2\left(T(B), \frac{d\mu(x)dt}{dt}\right)$. Therefore $g \in T_2^{p,\infty}$ and $\|g\|_{T_2^{p,\infty}} \leq C\|\mathcal{L}\|$. By part (1), we know

$$\mathcal{L}(W) = \int\limits_{X\times(0,\infty)} W(x,t)g(x,t)\frac{d\mu(x)dt}{t} \qquad W \in T_2^p.$$

This proves part (2) of Theorem 3, then Theorem 3 is proved. □

Similarly, we can prove the following corollary.

Corollary 1 *For $0 < p \leq 1$, we have*

$$\left(\widetilde{T_2^p}\right)^* = T_2^{p,\infty}.$$

By H^∞-functional calculas for L, we can prove (cf. [23])

$$Id = 4\int\limits_0^\infty \left(tL^{1/2}P_t^L\right)^2 \frac{dt}{t},$$

where the integral converges strongly in $L^2(\Omega)$. Therefore, for $f, g \in L^2(\Omega)$, we have

$$\int\limits_\Omega f(x)\overline{g(x)}dx = 4\int\limits_\Omega \int\limits_0^\infty \left(tL^{1/2}P_t^L f(y)\right)\left(\overline{tL^{1/2}P_t^L g(y)}\right)\frac{dydt}{t}. \tag{23}$$

By Theorem 3, we know

$$\left| 4 \int_\Omega \int_0^\infty \left(tL^{1/2} P_t^L f(y) \right) \overline{\left(tL^{1/2} P_t^L g(y) \right)} \frac{dydt}{t} \right| \le C \, \|s_L f\|_{L^p(\Omega)} \, \|Tg\|_{T_2^{p,\infty}}, \quad (24)$$

where Tg is defined as follows: when g is locally integrable on Ω with $g(y)(1+|y|)^{-d-1} \in L^2(\Omega)$, set

$$Tg(x) = \left(\sup_{x \in B} \frac{1}{|B \cap \Omega|^{2/p-1}} \int_{\widehat{B}} \left| tL^{1/2} P_t^L g(y) \right|^2 \frac{dydt}{t} \right)^{1/2},$$

where the supermum is taken over all the balls that contain x.

We define $\Lambda^L_{Z,\alpha}(\Omega)$ as the space of all functions in $\Lambda^L_\alpha(\mathbb{R}^d)$ that supported in $\overline{\Omega}$ equipped with the norm $\|f\|_{\Lambda^L_{Z,\alpha}(\Omega)} = \|f\|_{\Lambda^L_\alpha(\mathbb{R}^d)}$.

We give some properties of strongly Lipschitz domains of $\mathbb{R}^d$, whose proofs can be found in [1].

Lemma 7 *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^d$, then*

(a) *there is a constant $C > 0$ such that if B is a ball centered in Ω with $r_B \le diam(\Omega)$, we have $|B \cap \Omega| \ge C|B|$.*
(b) *there exists $\rho \in (0, \infty]$ such that if B is a ball with $2B \subset \Omega$, but $4B \cap \partial\Omega \ne \phi$ and $2r_B < \rho$, we can get a cube $\widetilde{B} \subset \Omega^c$ such that $|B| \sim |\widetilde{B}|$ and the distance from B to $\widetilde{B}$ is comparable to r_B. Furthermore, $\rho = \infty$ if Ω^c is unbounded.*

Now, we can prove

Lemma 8 *Let $\frac{d}{d+1} < p \le 1$, then under DBC and Ω^c unbounded, we have*

$$\left\| Tg \right\|_{T_2^{p,\infty}} \le C \left\| g \right\|_{\Lambda^L_{Z,d(1/p-1)}(\Omega)}$$

for all $g \in \Lambda^L_{Z,d(1/p-1)}(\Omega)$.

Proof For any ball $B(x_0, r)$ with $x_0 \in \Omega$ and $r \le diam(\Omega)$, we set $B_k = B(x_0, 2^k r)$ and

$$\begin{aligned} g &= (g - g(B_1 \cap \Omega)) \, \chi_{B_1 \cap \Omega} + (g - g(B_1 \cap \Omega)) \, \chi_{(B_1)^c \cap \Omega} + g(B_1 \cap \Omega) \\ &= g_1 + g_2 + g(B_1 \cap \Omega), \end{aligned}$$

where

$$g(B_1 \cap \Omega) = \frac{1}{|B_1 \cap \Omega|} \int_{B_1 \cap \Omega} g(x) dx.$$

For all $f \in L^2(\Omega)$, H^∞-functional calculas for L implies (cf. [23])

$$\left(\int_0^\infty \left\| D_t^L f \right\|^2_{L^2(\Omega)} \frac{dt}{t} \right)^{1/2} \le C \, \|f\|_{L^2(\Omega)}. \quad (25)$$

By (25), we get

$$\begin{aligned}
&\frac{1}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left|D_t^L g_1(y)\right|^2\frac{dydt}{t}\le\frac{C}{|B\cap\Omega|^{2/p-1}}\|g_1\|^2_{L^2(\Omega)}\\
&=\frac{C}{|B\cap\Omega|^{2/p-1}}\int\limits_{B_1\cap\Omega}|g(y)-g(B_1\cap\Omega)|^2\,dy\\
&\le\frac{C}{|B_1\cap\Omega|^{2/p-1}}\int\limits_{B_1\cap\Omega}|g(y)-g(B_1\cap\Omega)|^2\,dy\\
&\le C\,\|g\|_{\widetilde{\Lambda}^L_{Z,d(1/p-1)}(\Omega)},
\end{aligned}$$

where we use $\widetilde{\Lambda}^L_{Z,\alpha}(\Omega)$ to denote the space of functions which are integrable on $K\cap\Omega$ for all compact $K\subset\mathbb{R}^d$ such that $\|f\|_{\widetilde{\Lambda}^L_{Z,\alpha}(\Omega)}<\infty$ and

$$\|f\|_{\widetilde{\Lambda}^L_{Z,\alpha}(\Omega)}=\sup_{B\subset\Omega}\left(\frac{1}{|B\cap\Omega|^{2/p-1}}\int\limits_{B\cap\Omega}|g(y)-g(B\cap\Omega)|^2\,dy\right)^{1/2},$$

the supremum above is taken over all the balls $B(x,r)$ such that $x\in\Omega$ and $r<2diam(\Omega)$.

By Proposition 3, for $x\in B$,

$$\begin{aligned}
\left|D_t^L g_2(x)\right|&\le C\int\limits_{\Omega\setminus B_1}\frac{t}{(t^2+A|x-y|^2)^{(d+1)/2}}\left|g_2(y)\right|\,dy\\
&\le C\int\limits_{\Omega\setminus B_1}\frac{t}{|x_0-y|^{(d+1)}}\,|g(y)-g(B_1\cap\Omega)|\,dy\\
&\le C\sum_{k=1}^{\infty}\frac{t}{(2^k r)^{d+1}}\left(\int\limits_{\{y\in\Omega:2^k r<|x_0-y|\le 2^{k+1}r\}}|g(y)-g(B_{k+1}\cap\Omega)|\,dy\right.\\
&\qquad\left.+(2^k r)^d\,|g(B_{k+1}\cap\Omega)-g(B_1\cap\Omega)|\right)
\end{aligned}$$

If $diam(\Omega)=\infty$, then

$$\begin{aligned}
|g(B_{k+1}\cap\Omega)-g(B_k\cap\Omega)|&\le\frac{C}{|B_{k+1}\cap\Omega|}\int\limits_{B_{k+1}\cap\Omega}|g(y)-g(B_{k+1}\cap\Omega)|\,dy\\
&\le\frac{C}{|B_{k+1}\cap\Omega|^{1/2}}\left(\int\limits_{B_{k+1}\cap\Omega}|g(y)-g(B_{k+1}\cap\Omega)|^2\,dy\right)^{1/2}
\end{aligned}$$

$$= C\,|B_{k+1}\cap\Omega|^{1/p-1}\,\frac{1}{|B_{k+1}\cap\Omega|^{1/p-1/2}}\left(\int\limits_{B_{k+1}\cap\Omega}|g(y)-g(B_{k+1}\cap\Omega)|^2\,dy\right)^{1/2}$$
$$\le C\,|B_{k+1}\cap\Omega|^{1/p-1}\,\|g\|_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\,.$$

If $dima(\Omega)<\infty$, we assume that k_0 is the smallest integer such that $diam(\Omega)<2^{k_0}r$. Then $g(B_k\cap\Omega)=g(\Omega)$ and $\{y\in\Omega: 2^k r<|x_0-y|\le 2^{k+1}r\}=\phi$ for all $k\ge k_0$. When $k\le k_0$,

$$|g(B_k\cap\Omega)-g(B_{k-1}\cap\Omega)|\le\frac{C}{|B_k\cap\Omega|}\int\limits_{B_k\cap\Omega}|g(y)-g(B_k\cap\Omega)|\,dy$$
$$\le C\,|B_{k+1}\cap\Omega|^{1/p-1}\,\|g\|_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\,.$$

Therefore

$$\left|D_t^L g_2(x)\right|\le C\,\frac{t}{r^{1-d(1/p-1)}}\,\|g\|_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\sum_{k=1}^{\infty}2^{k(d(1/p-1)-1)}(1+k)$$
$$\le C\,\frac{t}{r^{1-d(1/p-1)}}\,\|g\|_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\,.$$

Then

$$\frac{1}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left|D_t^L g_2(x)\right|^2\frac{dxdt}{t}\le C\,\|g\|^2_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\,.$$

In the following, we will prove

$$\|g\|_{\widetilde{\Lambda}^L_{z,d(1/p-1)}(\Omega)}\le C\,\|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}\,. \tag{26}$$

If $B\subset\Omega$, then by Theorem 2 in [16],

$$\frac{1}{|B|}\int\limits_B|g(x)-g(B)|\,dx\le C\,\|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)}=C\,\|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}\,.$$

If $B\cap\partial\Omega\ne\phi$, since g vanishes outside Ω, by Lemma 7 (a),

$$\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}|g(x)-g(B\cap\Omega)|\,dx\le\frac{|B|}{|B\cap\Omega|}\frac{1}{|B|}\int\limits_B|g(x)-g(B)|\,dx$$
$$\le\frac{|B|}{|B\cap\Omega|}\,\|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)}\le C\,\|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}\,.$$

It remains to estimate the constant term. We first assume $r<\rho(x_0)$. Let $2^m r<\rho(x_0)\le 2^{m+1}r$, then we can prove

$$|g(B_1\cap\Omega)|\le C|B_{m+1}|^{1/p-1}\left(1+\log_2^{\frac{\rho(x_0)}{r}}\right)\|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}\,. \tag{27}$$

In face, if $B_2\subset\Omega$, let k_1 be the smallest integer such that $B_{k_1+1}\cap\partial\Omega\ne\phi$. When $m\le k_1$, by (cf. [17])

$$|g(B_1)-g(B)|\le C|B_1|^{1/p-1}\,\|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)}\,,$$

we get

$$\begin{aligned}|g(B_1)| &\le \sum_{i=1}^{m} |g(B_{i+1}) - g(B_i)| + |g(B_{m+1})| \\ &\le Cm|B_{m+1}|^{1/p-1} \|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)} + |B_{m+1}|^{1/p-1} \|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)} \\ &\le C|B_{m+1}|^{1/p-1} \left(1 + \log_2^{\frac{\rho(x_0)}{r}}\right) \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.\end{aligned}$$

When $m > k_1$, by Lemma 7 (b), we can choose $\widetilde{B}_1 \subset \Omega^c$ and $\widetilde{B}_2$ such that $|\widetilde{B}_1| = |B_{k_1+1}|$ and $\left(B_{k_1+1} \cup \widetilde{B}_1\right) \subset \widetilde{B}_2$, $|\widetilde{B}_2| \le C|\widetilde{B}_1|$, where C is independent of k_1. Let $2^{k_2} r < r_{\widetilde{B}_2} \le 2^{k_2+1} r$, if $k_2 \le m$,

$$\begin{aligned}\left|g(B_{k_1+1})\right| = \left|g(B_{k_1+1}) - g(\widetilde{B}_1)\right| &\le \frac{C}{|\widetilde{B}_2|} \int_{\widetilde{B}_2} \left|g(x) - g(B_{k_1+1})\right| dx \\ &\le \frac{C}{|\widetilde{B}_2|} \int_{\widetilde{B}_2} \left|g(x) - g(\widetilde{B}_2)\right| dx + \left|g(\widetilde{B}_2) - g(B_{k_1+1})\right| \\ &\le \frac{C}{|\widetilde{B}_2|} \int_{\widetilde{B}_2} \left|g(x) - g(\widetilde{B}_2)\right| dx + \frac{C}{|\widetilde{B}_2|} \int_{\widetilde{B}_2} \left|g(x) - g(\widetilde{B}_2)\right| \\ &\le C|B_{k_1+1}|^{1/p-1} \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.\end{aligned}$$

If $k_2 \ge m$, then

$$\begin{aligned}\left|g(B_{k_1+1})\right| = \frac{1}{\left|B_{k_1+1}\right|} \left| \int_{B_{k_1+1}} g(x)dx \right| &\le \frac{C}{|\widetilde{B}_2|} \int_{\widetilde{B}_2} |g(x)|\, dx \\ &\le C|B_{k_1+1}|^{1/p-1} \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.\end{aligned}$$

Therefore, when $k_1 < m$,

$$\begin{aligned}|g(B_1)| &\le \sum_{i=1}^{k_1} |g(B_{i+1}) - g(B_i)| + \left|g(B_{k_1+1})\right| \\ &\le Ck_1|B_{k_1+1}|^{1/p-1} \|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)} + |B_{k_1+1}|^{1/p-1} \|g\|_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)} \\ &\le C|B_{m+1}|^{1/p-1} \left(1 + \log_2^{\frac{\rho(x_0)}{r}}\right) \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.\end{aligned}$$

If $B_2 \cap \Omega \ne \phi$, taking a Whitney decomposition of $B_1 \cap \Omega$ with respect to $\partial\Omega$,

$$B_1 \cap \Omega = \bigcup_k B^k,$$

where B^k is a ball with $B^k \subset \Omega$ but $2B^k \cap \partial\Omega \ne \phi$. Then, similarly as the proof of the case $B_2 \subset \Omega$ (In fact, we just need to consider $k_1 = 1$), we can prove

$$\left|g(B^k)\right| \le C|B^k|^{1/p-1} \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)} \le C|B \cap \Omega|^{1/p-1} \|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.$$

Therefore, one has

$$|g(B_1\cap\Omega)|\le\sum_k\frac{|B^k|}{|B\cap\Omega|}|g(B^k)|\le C|B\cap\Omega|^{1/p-1}\,\|g\|_{\Lambda^L_{z,d(1/p-1)}(\Omega)}.$$

When $r<\rho(x_0)$, we have $\rho(x)\sim\rho(x_0)>r$ for any $x\in B(x_0,r)$, by Proposition 3 (c) and (27),

$$\begin{aligned}&\frac{1}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left|D_t^L\left(g(B_1\cap\Omega)\mathbf{1}\right)(x)\right|^2\frac{dx\,dt}{t} \qquad (28)\\ &=\frac{|g(B_1\cap\Omega)|^2}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left|\int\limits_\Omega D_t^L(x,y)\,dy\right|^2\frac{dx\,dt}{t}\\ &\le\frac{C\,|g(B_1\cap\Omega)|^2}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left(\frac{t}{\rho(x_0)}\right)^2\frac{dx\,dt}{t}\\ &\le C\,\frac{|B_{m+1}\cap\Omega|^{2/p-2}}{|B\cap\Omega|^{2/p-2}}\left(1+\log_2\frac{\rho(x_0)}{r}\right)^2\left(\frac{r}{\rho(x_0)}\right)^2\|g\|^2_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)}\\ &=C\left(1+\log_2\frac{\rho(x_0)}{r}\right)^2\left(\frac{r}{\rho(x_0)}\right)^{2-2d(1/p-1)}\|g\|^2_{\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)}\\ &\le C\,\|g\|^2_{\Lambda^L_{d(1/p-1)}(\Omega)}. \qquad (29)\end{aligned}$$

For $r\ge\rho(x_0)$, we have $|g(B_1\cap\Omega)|\le C\,|B_1\cap\Omega|^{1/p-1}\,\|g\|_{\Lambda^L_{d(1/p-1)}(\Omega)}$. Note that $\rho(x)\le Cr$ for any $x\in B(x_0,r)$, again by Proposition 3 (c), we get

$$\begin{aligned}&\frac{1}{|B\cap\Omega|^{2/p-1}}\int\limits_{T(B)}\left|D_t^L\left(g(B_1\cap\Omega)\mathbf{1}\right)(x)\right|^2\frac{dx\,dt}{t}\\ &\le\frac{|g(B_1\cap\Omega)|^2}{|B\cap\Omega|^{2/p-1}}\int_0^\infty\int_{B\cap\Omega}\left|\int\limits_\Omega D_t^L(x,y)\,dy\right|^2\frac{dx\,dt}{t}\\ &\le\frac{C\,|g(B_1\cap\Omega)|^2}{|B\cap\Omega|^{2/p-1}}\left(\int\limits_{B\cap\Omega}\int_0^{\rho(x)}\left(\frac{t}{\rho(x)}\right)^2\frac{dt}{t}\,dx+\int\limits_{B\cap\Omega}\int_{\rho(x)}^\infty\left(\frac{t}{\rho(x)}\right)^{-2}\frac{dt}{t}\,dx\right)\\ &\le C\,\|g\|^2_{\Lambda^L_{d(1/p-1)}(\Omega)}. \qquad (30)\end{aligned}$$

This proves Lemma 8. □

Let $\lambda^L_\alpha(\Omega)$ be the space of all the functions in $\lambda^L_\alpha(\mathbb{R}^d)$ such that compacted supported in $\overline{\Omega}$, then we can prove the following Lemma (cf. [4]).

Lemma 9 *If $\Omega\subset\mathbb{R}^d$ is a strongly Lipschitz domain, then the dual space of $\lambda^L_{d(1/p-1)}(\Omega)$ is $B^p_{r,L}(\Omega)$, where $B^p_{r,L}(\Omega)$ is the Banach complete space of $H^p_{r,L}(\Omega)$ and $\frac{d}{d+1}<p\le1$.*

Proof We have the following result (cf. [22]): let $\mathcal{B}$ be a Banach space and $\mathcal{B}^*$ the dual space of $\mathcal{B}$. Suppose $\mathcal{B}_1\subset\mathcal{B}$ is a closed subspace of $\mathcal{B}$, then $\mathcal{B}_1^*\cong\mathcal{B}^*/\mathcal{B}_1^\perp$, where $\mathcal{B}_1^\perp=\{\varphi\in\mathcal{B}^*:\varphi|_{\mathcal{B}_1}=0\}$. Now, we let $\mathcal{B}=\lambda^L_{d(1/p-1)}(\mathbb{R}^d)$ and $\mathcal{B}_1=\lambda^L_{d(1/p-1)}(\Omega)$, then by $\lambda^L_{d(1/p-1)}(\mathbb{R}^d)^*=B^p_L(\mathbb{R}^d)$, we get

$$\mathcal{B}_1^{\perp} = \{\varphi \in B_L^p(\mathbb{R}^d) : \varphi(f) = 0 \text{ for all } f \in \lambda^L_{d(1/p-1)}(\Omega)\}.$$

Therefore, for $\varphi \in \mathcal{B}_1^{\perp}$, there exists $g \in H_L^p(\mathbb{R}^d)$ such that for all $f \in \lambda^L_{d(1/p-1)}(\Omega)$,

$$\varphi(f) = \int_{\mathbb{R}^d} f(x)g(x)dx = \int_{\Omega} f(x)g(x) = 0.$$

This implies $\int_\Omega h(x)g(x)dx = 0$ for all $h(x)$ which is continuous and has compact supported in Ω, therefore $g(x) = 0$ for $x \in \Omega$. Thus $\mathcal{B}_1^{\perp} = \{g \in B_L^p(\mathbb{R}^d) : g|_\Omega = 0\}$. This proves that

$$\mathcal{B}^*/\mathcal{B}_1^{\perp} = \{\text{functions on } \Omega \text{ which can be extended to functions in } B_L^p(\mathbb{R}^d)\} \cong B^p_{r,L}.$$

Then Lemma 9 is proved. □

When f vanishes at infinity in a generalized sense, we can get the following version of Calderón reproducing formula.

Lemma 10 *If f vanishes at infinity in a generalized sense, then we have*

$$f = 4\int_0^{\infty} \left(D_t^L\right)^2 f \frac{dt}{t} \quad \text{in } (\Lambda_\alpha^L(\mathbb{R}^d))^*.$$

The meaning of above equality is

$$4\int_{\epsilon}^{A} \left(D_t^L\right)^2 f \frac{dt}{t} \to f \quad \text{when } \epsilon \to 0, \quad A \to \infty \quad \text{in } (\Lambda_\alpha^L(\mathbb{R}^d))^*.$$

Proof It is easy to know

$$\int_{\epsilon}^{A} \left(D_t^L\right)^2 f \frac{dt}{t} = \int_{\epsilon}^{\infty} \left(D_t^L\right)^2 f \frac{dt}{t} - \int_{A}^{\infty} \left(D_t^L\right)^2 f \frac{dt}{t} = I_1 - I_2.$$

Since f vanishes at infinity in a generalized sense, we get $\lim_{A\to\infty} I_2 = 0$. For any $\varphi \in \Lambda_\alpha^L$, we have

$$\begin{aligned}
&\left\langle \lim_{\epsilon\to 0} 4\int_{\epsilon}^{\infty} \left(D_t^L\right)^2 f \frac{dt}{t}, \varphi \right\rangle \\
&= \left\langle f, \lim_{\epsilon\to 0} 4\int_{\epsilon}^{\infty} \left(D_t^L\right)^2 \varphi \frac{dt}{t} \right\rangle \\
&= \langle f, \varphi \rangle .
\end{aligned}$$

The last equality follows from

$$\lim_{\epsilon\to 0} 4\int_{\epsilon}^{\infty} \left(D_t^L\right)^2 \frac{dt}{t} = I, \qquad \text{in } (\Lambda_\alpha^L(\mathbb{R}^d))^*,$$

where I is the identity operator in $(\Lambda_\alpha^L(\mathbb{R}^d))^*$.

This completes the proof of Lemma 10. □

Now, we can prove the final part of the main result.

Lemma 11 *Let Ω be a strongly Lipschtiz domain of $\mathbb{R}^d$ and Ω^c be unbounded. Then under DBC and f vanishes at infinity in a generalized sense, we have*

$$\left\| f \right\|_{H^p_{r,L}(\Omega)} \leq C \left\| s_L f \right\|_{L^p(\Omega)},$$

where $\frac{d}{d+1} < p \leq 1$.

Proof By Lemma 10,

$$f(x) = 4 \int_0^\infty (D_t^L)^2 f(x) \frac{dt}{t} \quad \text{in } (\Lambda^L_{Z,\alpha}(\Omega))^*.$$

Then (cf. [8])

$$f(x) = \sum_i \lambda_i \, \alpha_i(x) \quad \text{in } (\Lambda^L_{Z,\alpha}(\Omega))^*,$$

where $\alpha_i(x) = 4 \int_0^\infty D_t^L \, a_i(x,t) \frac{dt}{t}$, $i = 1, 2, \ldots$ and $a_i(x,t)$ are T_2^p-atoms. Moreover, we have

$$\inf \left\{ \sum_i \left|\lambda_i\right|^p \right\} \sim \|s_L f\|^p_{L^p(\Omega)},$$

where the infimum is taken over all decompositions $f(x) = \sum c_i \alpha_i$.

For all $i = 1, 2, \ldots$, we can prove that $\alpha_i(x)$ belongs to $L^2(\Omega)$ (cf. [16]) and $\|s_L \alpha_i\|_{L^p(\Omega)} \leq C$, in fact

$$\|\alpha_i\|_{L^2(\Omega)} \leq |B|^{1/2-1/p},$$

where B is the ball such that $supp\, a(x,t) \subset \widehat{B}$ and $a(x,t)$ is the T_2^p-atom associated to $\alpha(x)$.

By (23), (24), Lemmas 8 and 9, we get

$$\begin{aligned}
\|f\|_{H^p_{r,L}(\Omega)} &= \sup_{\substack{g \in \lambda^L_{Z,d(1/p-1)}(\Omega) \\ \|g\|_{\Lambda^L_{Z,d(1/p-1)}(\Omega)}=1}} \left| \int_\Omega f(x) g(x) dx \right| \\
&= \sup_{\substack{g \in \lambda^L_{Z,d(1/p-1)}(\Omega) \\ \|g\|_{\Lambda^L_{Z,d(1/p-1)}(\Omega)}=1}} \left| \int_\Omega \sum_i \lambda_i \, \alpha_i(x) g(x) dx \right| \\
&\leq \sum_i |\lambda_i| \sup_{\substack{g \in \lambda^L_{Z,d(1/p-1)}(\Omega) \\ \|g\|_{\Lambda^L_{Z,d(1/p-1)}(\Omega)}=1}} 4 \left| \int_{\Omega \times (0,\infty)} D_t^L \alpha_i(x) D_t^L g(x) \frac{dx dt}{t} \right|
\end{aligned}$$

$$\leq \sum_i |\lambda_i| \, ||s_L \alpha_i||L^p(\Omega) \sup_{\substack{g \in \lambda^L_{Z,d(1/p-1)}(\Omega) \\ \|g\|_{\Lambda^L_{Z,d(1/p-1)}(\Omega)}=1}} \|D_t^L g||_{T^\infty_{2,p}}$$

$$\leq C \sum_i |\lambda_i| \leq C \left(\sum_i |\lambda_i|^p \right)^{1/p} \leq C \|s_L f||_{L^p(\Omega)}.$$

Lemma 11 is proved. □

Proof of Theorem 1 Theorem 1 follows from Lemmas 1, 6 and 11. □

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