Atomic Decomposition of H^p Spaces Associated with Some Schrödinger Operators

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Abstract. Let $\{T_t\}_{t>0}$ be the semigroup generated by a Schrödinger operator $-A = \Delta - V$, where V is a nonnegative polynomial on $\mathbf{R}^d$. We say that f is in H^p_A associated with the operator A if the maximal function $\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|$ belongs to $L^p(\mathbf{R}^d)$. We characterize elements of the space H^p_A for $0 < p \leq 1$ by a special atomic decomposition.

1. Introduction. Let A be a Schrödinger operator on $\mathbf{R}^d$ which has the form

$$A = -\Delta + V, \tag{1.1}$$

where $V(x) = \sum_{\beta \leq \alpha} a_\beta x^\beta$ is a nonnegative nonzero polynomial on $\mathbf{R}^d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\in \mathbf{N}^d$, $\mathbf{N} = \{0, 1, 2, \ldots\}$. These operators were studied by a number of authors, cf. [Fe], [HN], [Z].

We say that f is in the space H^p_A if the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| \tag{1.2}$$

is in $L^p(\mathbf{R}^d)$, where $\{T_t\}_{t>0}$ is the semigroup generated by $-A$. The quasi-norm in H^p_A is defined by

$$\|f\|^p_{H^p_A} = \|\mathcal{M}f\|^p_{L^p(\mathbf{R}^d)}. \tag{1.3}$$

Let $\int_0^\infty \lambda \, dE_A(\lambda)$ be the spectral resolution of A. For a function φ from the

Indiana University Mathematics Journal ©, Vol. 47, No. 1 (1998)

Schwartz class $\mathcal{S}([0,\infty))$, $\varphi(0)\neq 0$, we define the maximal operator M_φ setting

$$M_\varphi f(x)=\sup_{t>0}|\varphi(tA)f(x)|, \tag{1.4}$$

where $\varphi(tA)f=\int_0^\infty \varphi(t\lambda)\,dE_A(\lambda)f$.

We shall show that the quasi-norms $\|\mathcal{M}f\|_{L^p}^p$ and $\|M_\varphi f\|_{L^p}^p$ are equivalent.

Our aim is to present an atomic characterization of the spaces H_A^p for $p\in(0,1]$.

Operators of the form (1.1) appear as images of certain homogeneous operators on homogeneous nilpotent Lie groups which, in fact, are differential operators if V is a sum of squares of polynomials, cf. e.g. [DHJ], [HJ], and references there. These methods are also used here. For $p=1$ similar results have been obtained in [DZ1].

We define an auxiliary function $m(x,V)$, see [Z], by

$$m(x,V)=\sum_{\beta\leq\alpha}|D^\beta V(x)|^{1/(|\beta|+2)}, \tag{1.5}$$

where $|\beta|=|(\beta_1,\beta_2,\dots,\beta_d)|=\beta_1+\beta_2+\cdots+\beta_d$.

Since V is a nonzero polynomial, there is a constant $c>0$ such that $c\leq m(x,V)$ for every $x\in\mathbf{R}^d$.

We set

$$\begin{aligned}\mathcal{B}_0&=\{x\in\mathbf{R}^d : c\leq m(x,V)<1\},\\ \mathcal{B}_n&=\{x\in\mathbf{R}^d : 2^{(n-1)/2}\leq m(x,V)<2^{n/2}\}\quad\text{for}\quad n=1,2,3\dots\ .\end{aligned} \tag{1.6}$$

We have $\mathbf{R}^d=\bigcup_{n=0}^\infty\mathcal{B}_n$. We will denote by $B(x,r)$ the ball in $\mathbf{R}^d$ with the center at x and radius r.

We say that a function a is an atom for the space H_A^p associated to a ball $B(x_0,r)$ if

$$\operatorname{supp} a\subset B(x_0,r), \tag{1.7}$$

$$\|a\|_{L^\infty}\leq(\operatorname{vol}B(x_0,r))^{-1/p}, \tag{1.8}$$

$$\text{if}\quad x_0\in\mathcal{B}_n,\quad\text{then}\quad r\leq 2^{1-n/2}, \tag{1.9}$$

$$\text{if}\quad x_0\in\mathcal{B}_n\ \text{and}\ \ r\leq 2^{-1-n/2},\quad\text{then}\quad\int x^\beta a(x)\,dx=0 \tag{1.10}$$

$$\text{for all}\quad |\beta|\leq d\left(\frac{1}{p}-1\right).$$

The atomic quasi-norm in the space H^p_A is defined by

$$\|f\|^p_{H^p_A \text{ atom}} = \inf\Big\{\sum_j |c_j|^p\Big\}, \tag{1.11}$$

where the infimum is taken over all decompositions $f = \sum_j c_j a_j$, with a_j being H^p_A atoms and c_j being scalars.

Our main result in this paper is as follows:

Theorem 1.12. *Let $\varphi \in \mathcal{S}([0,\infty))$, $\varphi(0) \neq 0$. Then for every $p \in (0,1]$, there is a constant $C > 0$ such that*

$$\|M_\varphi f\|_{L^p} \le C\|\mathcal{M} f\|_{L^p}, \tag{1.13}$$

$$\|\mathcal{M} f\|_{L^p} \le C\|f\|_{H^p_A \text{ atom}}, \tag{1.14}$$

$$\|f\|_{H^p_A \text{ atom}} \le C\|M_\varphi f\|_{L^p}. \tag{1.15}$$

Our spaces H^p_A are of a different nature than the classical Hardy spaces $H^p(\mathbf{R}^d)$, which may be thought of as the spaces $H^p_{-\Delta}(\mathbf{R}^d)$, Δ being the ordinary Laplacian on $\mathbf{R}^d$. It follows from Theorem 1.12 and properties of the sets $\mathcal{B}_n$, see Section 2, that $H^p(\mathbf{R}^d)$ is a proper subspace of the space H^p_A. Every element of H^p_A can be decomposed into atoms that are supported on small balls, but for certain atoms no moment condition is required. Actually our H^p_A atoms are scaled local atoms in the sense of Goldberg, cf. Section 8. The idea of the proof of Theorem 1.12 is based on the fact that the kernels of the operators T_t look like the classical heat kernels multiplied by $(1 + t^{1/2} m(x,V))^{-b}$, which have fast enough decay when t is large (cf. (3.18′)). The kernels of the operators $\varphi(tA)$ have a similar feature. If A is the Hermite operator, this type of decay of the kernels T_t can be read from Mehler's formula. In order to get appropriate estimates for the kernels $\varphi(tA)$, where A is a Schrödinger operator with a positive polynomial potential, we use the fact that A can be obtained as Π_P, where P is a regular kernel on a special nilpotent Lie group G and Π is a unitary representation of G, cf. Section 3. The theory of regular kernels on nilpotent Lie groups, which was developed by P. Głowacki, turns out to be crucial here, cf. Section 3.

Acknowledgments. The author is greatly indebted to Andrzej Hulanicki for his remarks. He also wishes to express his gratitude to the referee for his several helpful comments.

2. Resolution of identity associated with $\mathcal{B}_n$. The following two lemmas below present important properties of the sets $\mathcal{B}_n$.

Lemma 2.1. *There is a constant C such that for every $R > 2$ and every n, if $x \in \mathcal{B}_n$, then*

$$\{n' : B(x, 2^{-n/2}R) \cap \mathcal{B}_{n'} \neq \emptyset\} \subset [n - C\log_2 R,\ n + C\log_2 R].$$

Proof. Assume that $y \in B(x, 2^{-n/2}R) \cap \mathcal{B}_{n'}$, where $x \in \mathcal{B}_n$. By the definition of $\mathcal{B}_{n'}$ there is $\beta \leq \alpha$ such that $(1/C)2^{(n'-1)(|\beta|+2)/2} \leq |D^\beta V(y)|$. Applying the Taylor formula and the fact that $x \in \mathcal{B}_n$, we obtain

$$\frac{1}{C}2^{(n'-1)(|\beta|+2)/2} \leq \Big|\sum_{\gamma \leq \alpha} \frac{1}{\gamma!} D^{\gamma+\beta}V(x)(y-x)^\gamma\Big| \leq C2^{(|\beta|+2)n/2}R^{|\alpha|},$$

which implies $n' \leq n + C\log_2 R$. In the same manner we can see that $n \leq n' + C\log_2 R$. □

Lemma 2.2. *There is a constant C and a collection of balls $B_{(n,k)} = B(x_{(n,k)}, 2^{1-n/2})$, $n = 0,1,2,\ldots$, $k = 1,2,\ldots$, such that $x_{(n,k)} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B(x_{(n,k)}, 2^{-n/2})$, and $\#\{(n',k') : B(x_{(n,k)},\ R2^{-n/2}) \cap B(x_{(n',k')},\ R2^{-n'/2}) \neq \emptyset\} \leq R^C$ for every (n,k) and $R \geq 2$.*

Proof. For fixed n let $B(x_{(n,k)}, 2^{-2-n/2})$ be a countable sequence of balls such that $x_{(n,k)} \in \mathcal{B}_n$, $B(x_{(n,k)}, 2^{-2-n/2}) \cap B(x_{(n,k')}, 2^{-2-n/2}) = \emptyset$ for $k \neq k'$, and $\mathcal{B}_n \subset \bigcup_k B(x_{(n,k)}, 2^{-n/2})$. It is now easy to check, using Lemma 2.1, that the family of balls $B(x_{(n,k)}, 2^{1-n/2})$, $n = 0,1,\ldots$, $k = 1,2,\ldots$ satisfies the conclusion of Lemma 2.2. □

As a consequence of Lemma 2.2, we obtain the following result:

Lemma 2.3. *There are nonnegative functions $\psi_{(n,k)}$ such that*

$$\psi_{(n,k)} \in C_c^\infty(B(x_{(n,k)}, 2^{1-n/2})), \tag{2.4}$$

$$\sum_{(n,k)} \psi_{(n,k)}(x) = 1, \tag{2.5}$$

$$\Big\|\frac{\partial^{|\beta|}}{\partial x^\beta}\psi_{(n,k)}\Big\|_{L^\infty} \leq C_\beta 2^{|\beta|n/2}. \tag{2.6}$$

3. Functional calculus of operators on homogeneous groups and Schrödinger operators. Let G be a homogeneous nilpotent group equipped with a family of dilations δ_t (cf. [FS]). A distribution P on G is called a regular

kernel of order $r > 0$, if P coincides with a smooth function away from the origin and

$$\langle P, f \circ \delta_t \rangle = t^r \langle P, f \rangle. \tag{3.1}$$

It was proved in [D2] that for every Schrödinger operator of the form (1.1) there exist a homogeneous nilpotent Lie group G, a unitary representation Π of G, and a regular symmetric kernel P of order 2 such that

$$\Pi_P = A. \tag{3.2}$$

We shall denote by the same letter P the convolution operator $f \mapsto f * P$. The kernel P satisfies the following maximal subelliptic estimates proved by P. Głowacki (cf. [G], [D2]); for every left-invariant homogeneous differential operator ∂ on G and every $N \geq |\partial|/2$ there is a constant $C > 0$ such that

$$\|\partial f\|_{L^2(G)} \leq C \|(1+P)^N f\|_{L^2(G)}, \quad \text{for} \quad f \in C_c^\infty(G), \tag{3.3}$$

where $|\partial|$ is the degree of homogeneity of ∂.

Topologically G can be written as $G = \mathcal{X} \oplus \mathcal{Y} = \mathbf{R}^d \oplus \mathbf{R}^D$, where $D = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_d + 1)$. The construction of the group G and the representation Π is such that for every multi-index $\beta \in \mathbf{N}^d$ there is a left-invariant differential operator ∂ on G such that

$$\frac{\partial^{|\beta|}}{\partial x^\beta} = \Pi_\partial. \tag{3.4}$$

Let $\{S_t\}_{t>0}$ be the semigroup of linear operators on $L^2(G)$ generated by the essentially self-adjoint operator $-P$. The estimates (3.3) and the homogeneity of P imply that the semigroup $\{S_t\}_{t>0}$ is of the form

$$S_t f = f * q_t, \quad q_t(g) = t^{-Q/2} q_1(\delta_{t^{-1/2}} g), \quad \text{where } q_t \in C^\infty(G) \cap L^2(G),$$

Q denotes the homogeneous dimension of G. Moreover, the following estimates for q_t hold (cf. [G], [D2]): for every left-invariant homogeneous differential operator ∂ on G there is a constant $C > 0$ such that

$$|\partial q_t(g)| \leq C t (t^{1/2} + |g|)^{-Q-|\partial|-2}, \tag{3.5}$$

where $|g|$ is a fixed homogeneous norm on G and $|\partial|$ is the degree of homogeneity of ∂.

For a positive integer k let $W^k(G)$ denote the set of all functions $\mathbf{F} \in C^k(G)$ such that

$$\sup_{g\in G} |\partial \mathbf{F}(g)(1+|g|)^k| < \infty$$

for every left-invariant differential operator ∂ on G of degree $\le k$.

Similarly, we define $W^k(\mathbf{R}^d \times \mathbf{R}^D)$ to be the set of all functions $\tilde{F} = \tilde{F}(x,\xi) \in C^k(\mathbf{R}^d \times \mathbf{R}^D)$ such that

$$\sup_{(x,\xi)\in \mathbf{R}^d\times\mathbf{R}^D} \left| \frac{\partial^{|\beta|}}{\partial x^\beta} \frac{\partial^{|\beta'|}}{\partial \xi^{\beta'}} \tilde{F}(x,\xi)(1+|x|+|\xi|)^k \right| < \infty$$

for all $\beta \in \mathbf{N}^d$, $\beta' \in \mathbf{N}^D$ such that $|\beta| + |\beta'| \le k$.

Obviously, cf. [FS], for every $k > 0$ there exists k' such that $W^{k'}(G) \subset W^k(\mathbf{R}^d \times \mathbf{R}^D)$ and $W^{k'}(\mathbf{R}^d \times \mathbf{R}^D) \subset W^k(G)$.

For an integer $m > 0$ let us denote by $\mathcal{S}_0^m([0,\infty))$ the subspace of all functions φ from $\mathcal{S}([0,\infty))$ such that

$$\frac{d^j}{d\lambda^j}\varphi(0^+) = 0 \quad \text{for } j = 1,2,3,\dots m.$$

We set $\mathcal{S}_0^\infty([0,\infty)) = \bigcap_{m=1}^\infty \mathcal{S}_0^m([0,\infty))$.

Let $\int_0^\infty \lambda\, dE_P(\lambda)$ be the spectral resolution of the positive definite operator P. The theorem below was actually proved in [D1].

Theorem 3.6 (cf. [D1])**.** *For every $k > 0$ there is an integer $m > 0$ such that for $\varphi \in \mathcal{S}_0^m([0,\infty))$ there is a function $\mathbf{F}^{[\varphi]} \in W^k(G)$ such that*

$$\varphi(P)f = f * \mathbf{F}^{[\varphi]},$$

where $\varphi(P)f = \int_0^\infty \varphi(\lambda)\, dE_P(\lambda) f$.

Moreover, for every left-invariant differential operator ∂ on G of degree $\le k$ there is a constant C such that for $\varphi \in \mathcal{S}_0^m([0,\infty))$ we have

$$|\partial \mathbf{F}^{[\varphi]}(g)|(1+|g|)^k \le C \sum_{j=0}^m \sup_{\lambda\ge 0} \left\{ (1+\lambda)^m \left| \frac{d^j}{d\lambda^j}\varphi(\lambda) \right| \right\}.$$

Corollary 3.7. *If $\varphi \in \mathcal{S}_0^\infty([0,\infty))$, then there is a Schwartz class function $\mathbf{F}^{[\varphi]}$ on G such that*

$$\varphi(P)f = f * \mathbf{F}^{[\varphi]}.$$

Moreover, $\varphi(tP)f = f * \mathbf{F}_t^{[\varphi]}$, where $\mathbf{F}_t^{[\varphi]}(g) = t^{-Q/2}\mathbf{F}^{[\varphi]}(\delta_{t^{-1/2}}g)$.

We shall need the following result:

Proposition 3.8. *Assume that $\psi \in \mathcal{S}([0,\infty))$, $0 \notin \operatorname{supp}\psi$. Let $\mathbf{F}^{[\psi]}$ be the convolution kernel of $\psi(P)$. Then*

$$\int_G x^\beta \mathbf{F}^{[\psi]}(x)\,dx = 0 \tag{3.9}$$

for every multi-index $\beta \in \mathbf{N}^{d+D}$.

Proof. For a multi-index β let N be a positive integer such that $N > \|\beta\| + Q$, where $\|\beta\|$ is the homogeneous degree of the monomial x^β on G. Then by the spectral theorem

$$\mathbf{F}^{[\psi]}(x) = P^N \mathbf{F}^{[\omega]}(x), \tag{3.10}$$

where $\omega(\lambda) = \lambda^{-N}\psi(\lambda)$. Observe that the operator P^N is a convolution with a regular kernel of order $2N$. Therefore, $x^\beta * P^N$ is well defined, and

$$\int_G \mathbf{F}^{[\psi]}(x)x^\beta\,dx = \int_G (\mathbf{F}^{[\omega]} * P^N)(x)x^\beta\,dx = \int_G \mathbf{F}^{[\omega]}(x)(x^\beta * P^N)\,dx\,.$$

Note that $x^\beta * P^N$ is a C^∞ function on G homogeneous of degree $\|\beta\| - 2N < 0$. Thus $x^\beta * P^N \equiv 0$, which implies (3.9). □

The properties of the kernels $\mathbf{F}_t^{[\varphi]}$ of the operators $\varphi(tP)$ on the group G allow us to obtain suitable estimates for the integral kernels $\Phi_t(x,w)$ of the operators $\varphi(tA)$ on $\mathbf{R}^d$. One can deduce from the equality $\Pi_P = A$ that

$$\Pi_{\varphi(tP)} = \varphi(tA) \quad \text{for } \varphi \in \mathcal{S}([0,\infty)). \tag{3.11}$$

The construction of G, Π, and P is such that if $\mathbf{F} \in L^1(G)$, then the integral kernel $F(x,w)$ of the of the operator $\Pi_{\mathbf{F}}$ on $\mathbf{R}^d$ is given by

$$\begin{aligned} F(x,w) &= \tilde{F}(w-x, V(x), \dots, D^\beta V(x), \dots), \\ \tilde{F}(x,\xi) &= (\mathcal{F}_{\mathcal{Y}}\mathbf{F})(x,\xi), \end{aligned} \tag{3.12}$$

where $\mathcal{F}_{\mathcal{Y}}\mathbf{F}$ is the partial Fourier transform of $\mathbf{F}$ with respect to $\mathcal{Y}$ variables (cf. [D2], [DHJ]). Consequently, if $\mathbf{F}_t(g) = t^{-Q/2}\mathbf{F}(\delta_{t^{-1/2}}g)$, then the integral kernel $F_t(x,w)$ of the operator $\Pi_{\mathbf{F}_t}$ is of the form

$$F_t(x,w) = t^{-d/2}\tilde{F}\left(\frac{w-x}{t^{1/2}}, tV(x), \dots, t^{(|\beta|+2)/2}D^\beta V(x), \dots\right). \tag{3.13}$$

For a function $\varphi \in \mathcal{S}_0^m([0,\infty))$ we denote by $\mathbf{F}_t^{[\varphi]}$ the convolution kernel of the operator $\varphi(tP)$ on G, and by $\Phi_t(x,w)$ the integral kernel of the operator $\varphi(tA) = \Pi_{\mathbf{F}_t^{[\varphi]}}$. By (3.12), (3.13), Theorem 3.6, Corollary 3.7, and the homogeneity of P, we have the following:

Corollary 3.14. *For every integer $k > 0$ there exists $m > 0$ such that if $\varphi \in \mathcal{S}_0^m([0,\infty))$, then the kernels Φ_t and $\mathbf{F}^{[\varphi]} = \mathbf{F}_1^{[\varphi]}$ are related by*

$$\Phi_t(x,w) = t^{-d/2}\tilde{\Phi}\left(\frac{w-x}{t^{1/2}}, tV(x), \dots, t^{(|\beta|+2)/2}D^\beta V(x), \dots\right), \tag{3.15}$$

where

$$\tilde{\Phi}(x,\xi) = (\mathcal{F}_{\mathcal{Y}}\mathbf{F}^{[\varphi]})(x,\xi) \in W^k(\mathbf{R}^d \times \mathbf{R}^D). \tag{3.16}$$

If, moreover, $\varphi \in \mathcal{S}_0^\infty([0,\infty))$, then

$$\tilde{\Phi}(x,\xi) \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^D).$$

Here and subsequently, $\mathcal{S}(\mathbf{R}^d \times \mathbf{R}^D)$ denotes the Schwartz class of functions on $\mathbf{R}^d \times \mathbf{R}^D$.

Let us denote by $T_t(x,w)$ the integral kernels of the semigroup $T_t = \Pi_{q_t}$.

Proposition 3.17. *For every $b > 0$ there is a constant C_b such that*

$$\begin{aligned} 0 &\le T_t(x,w) \\ &\le C_b t^{-d/2}(1 + t^{-1/2}|x-w|)^{-b} \prod_{\beta \le \alpha} (1 + t^{(|\beta|+2)/2}|D^\beta V(x)|)^{-b}. \end{aligned} \tag{3.18}$$

For every multiindex $\beta \in \mathbf{N}^d$ there is a constant $C > 0$ such that

$$\left|\frac{\partial^{|\beta|}}{\partial x^\beta} T_t(x,w)\right| \le C t^{-(d+|\beta|)/2}(1 + t^{-1/2}|x-w|)^{-d-2-|\beta|}. \tag{3.19}$$

Proof. The estimate (3.18) was proved in [D2, Proposition 3.17]. To prove (3.19) we use (3.5), (3.12), and (3.4). □

It was pointed out by the referee that the estimate (3.18) can be written as

$$0 \le T_t(x,w) \le C_b t^{-d/2}(1 + t^{-1/2}|x-w|)^{-b}(1 + t^{1/2}m(x,V))^{-b'}. \tag{3.18$'$}$$

Assume that $\psi \in \mathcal{S}([0,\infty))$ be such that $0 \notin \operatorname{supp}\psi$. Similarly, we denote by $\mathbf{F}_t^{[\psi]}$ the convolution kernel of the operator $\psi(tP)$ and by $\Psi_t(x,w)$ the integral kernel of the operator $\psi(tA)$. Obviously, by Corollary 3.14, we have

$$\Psi_t(x,w) = t^{-d/2}\tilde{\Psi}\left(\frac{w-x}{t^{1/2}}, tV(x), \dots, t^{(|\beta|+2)/2}D^\beta V(x), \dots\right), \tag{3.15$'$}$$

where

$$\tilde{\Psi}(x,\xi) = (\mathcal{F}_{\mathcal{Y}}\mathbf{F}^{[\psi]})(x,\xi) \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^D). \tag{3.16$'$}$$

The corollary below follows from Proposition 3.8.

Corollary 3.20. *For every multi-index $\gamma \in \mathbf{N}^D$ and $\beta \in \mathbf{N}^d$ we have*

$$\int_{\mathbf{R}^d} \frac{\partial^{|\gamma|}}{\partial\xi^\gamma}\tilde{\Psi}(u-w,\xi)\Big|_{\xi=0} w^\beta\, dw = 0. \tag{3.21}$$

4. Tangential maximal functions associated to A.

Lemma 4.1. *Assume that $m \in \{1,2,3,\dots\}$ and θ, $\varphi \in \mathcal{S}_0^m([0,\infty))$, $\varphi(0) \neq 0$. Then there are $\psi^{(0)} \in \mathcal{S}_0^m([0,\infty))$ and $\psi \in C_c^\infty(0,1)$ such that*

$$1 = \psi^{(0)}(\lambda)\varphi(\lambda) + \sum_{j=1}^{\infty}\psi(2^{-j}\lambda)\varphi(2^{-j}\lambda) \quad \text{for } \lambda \geq 0, \tag{4.2}$$

and, consequently

$$\theta(\lambda) = \psi^{(0)}(\lambda)\theta(\lambda)\varphi(\lambda) + \sum_{j=1}^{\infty}\psi(2^{-j}\lambda)\theta(\lambda)\varphi(2^{-j}\lambda) \quad \text{for } \lambda \geq 0.$$

Proof. Since $\varphi(0) \neq 0$ there exists a positive integer k and a constant $c > 0$ such that $|\varphi(\lambda)| > c$ for $\lambda \in [0, 2^{-k}]$. Therefore there is a function $\psi \in C_c^\infty(2^{-k-7}, 2^{-k-5})$ such that $\sum_{j=-\infty}^{\infty}\psi(2^{-j}\lambda)\varphi(2^{-j}\lambda) = 1$ for $\lambda > 0$. Set

$$\eta(\lambda) = \sum_{j=1}^{\infty}\psi(2^{-j}\lambda)\varphi(2^{-j}\lambda).$$

Obviously $\eta(\lambda) = 1$ for $\lambda > 2^{-k-1}$. Put

$$\psi^{(0)}(\lambda) = \begin{cases} \dfrac{1-\eta(\lambda)}{\varphi(\lambda)} & \text{for } \lambda \in [0, 2^{-k}] \\ 0 & \text{for } \lambda > 2^{-k}. \end{cases}$$

It is easy to verify that $\psi^{(0)} \in \mathcal{S}_0^m([0,\infty))$ and (4.2) holds. □

For functions $\psi^{(0)}$, ψ, and θ from Lemma 4.1 let us denote by $K_t^{(0)}(x,y)$ and $K_{2^{-j}t}(x,y)$ the integral kernels of the operators $\psi^{(0)}(tA)\theta(tA)$ and $\psi(2^{-j}tA)\theta(tA)$ respectively. The following lemma can be deduced from (3.4), (3.13), Theorem 3.6 and Corollary 3.14.

Lemma 4.3. *For every M, M'', $k > 0$ there there exists $m > 0$ such that if θ, φ, $\psi^{(0)}$, $\psi \in \mathcal{S}_0^m([0,\infty))$ are from Lemma 4.1. Then there exists a constant $C > 0$ such that*

$$\left|\frac{\partial^{|\beta|}}{\partial x^\beta} K_t^{(0)}(x,y)\right| \le C t^{-(d+|\beta|)/2} \left(1 + \frac{|x-y|}{t^{1/2}}\right)^{-M}, \tag{4.4}$$

$$\left|\frac{\partial^{|\beta|}}{\partial x^\beta} K_{2^{-j}t}(x,y)\right| \le C(2^{-j}t)^{-(d+|\beta|)/2} 2^{-jM''} \left(1 + \frac{|x-y|}{(2^{-j}t)^{1/2}}\right)^{-M}, \tag{4.5}$$

for every β, $|\beta| \le k$.

Remark. One should not confuse the functions $\psi^{(0)}$, ψ that are defined on $[0,\infty)$ with the functions $\psi_{(n,k)}$ (from Lemma 2.3) that are defined on $\mathbf{R}^d$.

For a positive real number N and $\varphi \in \mathcal{S}([0,\infty))$ we define an analogue of Peetre tangential maximal operator $M_{\varphi,N}^*$ by

$$M_{\varphi,N}^* f(x) = \sup_{t>0,\ y\in\mathbf{R}^d} \{|\varphi(tA)f(y)|(1 + t^{-1/2}|y-x|)^{-N}\}. \tag{4.6}$$

The operators $M_{\varphi,N}^*$ have many of the properties of classical maximal operators. We present some of them which we shall need later. For the convenience of the reader we provide the proofs.

Lemma 4.7. *For every $N > 0$ there exists $m > 0$ such that if θ, $\varphi \in \mathcal{S}_0^m([0,\infty))$ such that $\varphi(0) \ne 0$, then there exists a constant C such that*

$$M_\theta f(x) \le C M_{\varphi,N}^* f(x). \tag{4.8}$$

Proof. Lemmas 4.1 and 4.3 imply

$$\begin{aligned}
M_\theta f(x) &= \sup_{t>0}\Big|\psi^{(0)}(tA)\theta(tA)\varphi(tA)f(x)+\sum_{j=1}^{\infty}\psi(2^{-j}tA)\theta(tA)\varphi(2^{-j}tA)f(x)\Big| \\
&\le \sup_{t>0}\int |K_t^{(0)}(x,y)\varphi(tA)f(y)|\,dy \\
&\qquad +\sup_{t>0}\sum_{j=1}^{\infty}\int |K_{2^{-j}t}(x,y)\varphi(2^{-j}tA)f(y)|\,dy \\
&\le \sup_{t>0} C\sum_{j=0}^{\infty}2^{-jM''}\int (2^{-j}t)^{-d/2}\left(1+\frac{|x-y|}{(2^{-j}t)^{1/2}}\right)^{-M} \\
&\qquad \times\left(1+\frac{|x-y|}{(2^{-j}t)^{1/2}}\right)^{N} M^*_{\varphi,N}f(x)\,dy,
\end{aligned}$$

which gives (4.8). □

Lemma 4.9. *For every $N>0$ there exists $m>0$ such that if $\varphi\in\mathcal{S}_0^m([0,\infty))$, $\varphi(0)\neq 0$, then there is a constant $C>0$ such that*

$$\left|\frac{\partial}{\partial x_i}\varphi(tA)f(x)\right| \le Ct^{-1/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{N} M^*_{\varphi,N}f(w). \tag{4.10}$$

Proof. Applying Lemmas 4.1 and 4.3 to $\theta=\varphi$, we obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial x_i}\varphi(tA)f(x)\right| \\
&\le \int\left|\frac{\partial}{\partial x_i}K_t^{(0)}(x,y)\varphi(tA)f(y)\right|dy \\
&\qquad +\sum_{j=1}^{\infty}\int\left|\frac{\partial}{\partial x_i}K_{2^{-j}t}(x,y)\varphi(2^{-j}tA)f(y)\right|dy \\
&\le C\sum_{j=0}^{\infty}\int (2^{-j}t)^{-(d+1)/2}2^{-jM''}\left(1+\frac{|x-y|}{(2^{-j}t)^{1/2}}\right)^{-M} \\
&\qquad \times\left(1+\frac{|w-y|}{(2^{-j}t)^{1/2}}\right)^{N}\left(1+\frac{|w-y|}{(2^{-j}t)^{1/2}}\right)^{-N}|\varphi(2^{-j}tA)f(y)|\,dy \quad \le
\end{aligned}$$

$$\le C\sum_{j=0}^{\infty}\int t^{-(d+1)/2}2^{-jM''}2^{j(d+N+1)/2}\left(1+\frac{|x-y|}{t^{1/2}}\right)^{-M}$$
$$\times\left(1+\frac{|x-y|}{t^{1/2}}\right)^{N}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{N}M^*_{\varphi,N}f(w)\,dy$$
$$\le Ct^{-1/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{N}M^*_{\varphi,N}f(w). \qquad \square$$

Proposition 4.11. *For every $N>0$ there is $m>0$ such that for every function $\varphi\in\mathcal{S}_0^m([0,\infty))$, $\varphi(0)\neq 0$, there is a constant $C>0$ such that*

$$M^*_{\varphi,N}f(x)\le C[\mathcal{M}_{[H-L]}(M_\varphi f)^r(x)]^{1/r}, \tag{4.12}$$

where $r=d/N$ and $\mathcal{M}_{[H-L]}$ is the classical Hardy-Littlewood maximal operator.

Proof. Fix $0<\delta_1<1$ and set $\delta=t^{1/2}\delta_1$. By the Mean Value Theorem we get

$$|\varphi(tA)f(x-u)|$$
$$\le C\delta^{-d/r}\left(\int_{|x-u-y|\le\delta}|\varphi(tA)f(y)|^r\,dy\right)^{1/r}+C\delta\sup_{|x-y-u|\le\delta}|\nabla\varphi(tA)f(y)|$$
$$\le C\delta^{-d/r}(\delta+|u|)^{d/r}[\mathcal{M}_{[H-L]}|\varphi(tA)f|^r(x)]^{1/r}$$
$$+\ C\delta\sup_{|x-u-y|\le\delta}|\nabla\varphi(tA)f(y)|\left(1+\frac{|x-y|}{t^{1/2}}\right)^{-N}\left(1+\frac{|x-y|}{t^{1/2}}\right)^{N}.$$

Lemma 4.9 leads to

$$|\varphi(tA)f(x-u)|$$
$$\le C\delta^{-d/r}(\delta+|u|)^{d/r}[\mathcal{M}_{[H-L]}|\varphi(tA)f|^r(x)]^{1/r}+Ct^{-1/2}\delta M^*_{\varphi,N}f(x)\left(1+\frac{\delta}{t^{1/2}}+\frac{|u|}{t^{1/2}}\right)^{N}$$
$$\le C\delta_1^{-d/r}\left(1+\frac{|u|}{t^{1/2}}\right)^{d/r}[\mathcal{M}_{[H-L]}|\varphi(tA)f|^r(x)]^{1/r}+C\delta_1M^*_{\varphi,N}f(x)\left(1+\delta_1+\frac{|u|}{t^{1/2}}\right)^{N}.$$

Therefore,

$$|\varphi(tA)f(x-u)|\left(1+\frac{|u|}{t^{1/2}}\right)^{-N}$$
$$\le C\delta_1^{-d/r}[\mathcal{M}_{[H-L]}|\varphi(tA)f|^r(x)]^{1/r}+C\delta_1 M^*_{\varphi,N}f(x).$$

Taking δ_1 sufficiently small, we get (4.12). □

Proposition 4.13. *For every $p\in(0,1]$ there exists $m>0$ such that if $\theta,\ \varphi\in S_0^m([0,\infty))$, $\varphi(0)\neq 0$, then there exists a constant $C>0$ such that*

$$\|M_\theta f\|_{L^p}\le C\|M_\varphi f\|_{L^p}. \tag{4.14}$$

Proof. Fix $N>0$ such that $r=d/N<p$. Let $m>0$ be such that Lemma 4.7 and Proposition 4.11 hold. By Lemma 4.7 it suffices to show that

$$\|M^*_{\varphi,N}f\|_{L^p}\le C\|M_\varphi f\|_{L^p}. \tag{4.15}$$

By virtue of Proposition 4.11, we obtain

$$\begin{aligned}\|M^*_{\varphi,N}f\|_{L^p}&\le C\|[\mathcal{M}_{[H-L]}(M_\varphi f)^r]^{1/r}\|_{L^p}\\&=C\|\mathcal{M}_{[H-L]}(M_\varphi f)^r\|^{1/r}_{L^{p/r}}\\&\le C\|(M_\varphi f)^r\|^{1/r}_{L^{p/r}}=C\|M_\varphi f\|_{L^p}.\end{aligned}$$

5. Proof of (1.13)**.** For $p\in(0,1]$ let $m>0$ be large enough such that Proposition 4.13 holds. If $\varphi\in\mathcal{S}([0,\infty))$, then there are constants $c_1,c_2,\dots,c_{m+1}$ such that $\theta(\lambda)=\varphi(\lambda)-\sum_{n=1}^{m+1}c_ne^{-n\lambda}\in\mathcal{S}_0^m([0,\infty))$. Therefore

$$\|M_\varphi f\|_{L^p}\le C(\|M_\theta f\|_{L^p}+\|\mathcal{M}f\|_{L^p}).$$

By Proposition 4.13 it suffices to show that there exists a function $\varphi\in\mathcal{S}_0^m([0,\infty))$, $\varphi(0)\neq 0$, such that

$$\|M_\varphi f\|_{L^p}\le C\|\mathcal{M}f\|_{L^p}. \tag{5.1}$$

Such a function can be constructed as a linear combination of functions $e^{-n\lambda}$,

that is, there are constants $d_1, d_2, \dots, d_{m+1}$ such that

$$\varphi(\lambda) = \sum_{n=1}^{m+1} d_n e^{-n\lambda} \in \mathcal{S}_0^m([0,\infty)) \quad \text{and } \varphi(0) = 1. \tag{5.2}$$

Obviously for φ of the form (5.2) the inequality (5.1) is satisfied.

6. Proof of (1.14). It suffices to show that there is a constant $C > 0$ such that

$$\|\mathcal{M}a\|_{L^p}^p \le C \tag{6.1}$$

for every H_A^p atom a. Let a be an H_A^p atom associated to a ball $B(x_0, r)$. Assume that $x_0 \in \mathcal{B}_n$. By the definition of the atoms $r \le 2^{1-n/2}$. It follows from (1.8) and the estimates (3.18) that

$$\mathcal{M}a(x) \le C r^{-d/p} \quad \text{for } x \in B^*(x_0, r) = B(x_0, 2r), \tag{6.2}$$

and, consequently,

$$\int_{B(x_0,2r)} (\mathcal{M}a(x))^p \, dx \le C. \tag{6.3}$$

In order to show that

$$\int_{B(x_0,2r)^c} (\mathcal{M}a(x))^p \, dx \le C\,, \tag{6.4}$$

we consider two cases:

$\ggg$ CASE 1: $r \le 2^{-1-n/2}$

Then, by definition, a satisfies the moment conditions (1.10). Let τ be the smallest integer $> d(1/p - 1)$. For fixed $x \in B(x_0, 2r)^c$ let us consider the Taylor expansion of the function $w \mapsto T_t(x, w)$ at x_0. According to (3.19) and the equality $T_t(x, w) = T_t(w, x)$, we get

$$
\begin{aligned}
|T_t a(x)| &= \left| \int a(w) T_t(x,w)\, dw \right| \\
&= \left| \int a(w) [T_t(x,w) - \sum_{|\gamma|<\tau} \frac{1}{\gamma!} \left(\frac{\partial^{|\gamma|}}{\partial w^\gamma} T_t \right)(x,w) \Big|_{w=x_0} (w-x_0)^\gamma]\, dw \right| \\
&\le C t^{-\tau/2} t^{-d/2} \int |a(w)| \left(1 + \frac{|x-x_0|}{t^{1/2}} \right)^{-d-\tau} |w-x_0|^\tau\, dw \\
&\le C |x-x_0|^{-d-\tau} r^{\tau+d} r^{-d/p}.
\end{aligned}
$$

Therefore

$$
\int_{B(x_0,2r)^c} (\mathcal{M} a(x))^p\, dx \le C r^{p\tau+pd} r^{-d} \int_{B(x_0,2r)^c} |x-x_0|^{-(d+\tau)p}\, dx \le C. \tag{6.5}
$$

⋙ Case 2: $2^{-1-n/2} < r \le 2^{-n/2}$

Note that in this case no moment condition on a is required. For $0 < t \le 2^{-n}$ and $x \in B(x_0,2r)^c$, we have

$$
\begin{aligned}
|T_t a(x)| &\le C \int |a(w)| T_t(x,w)\, dw \\
&\le C t^{-d/2} \int_{B(x_0,r)} r^{-d/p} \left(1 + \frac{|x-w|}{t^{1/2}} \right)^{-M} dw \\
&\le C t^{-d/2} r^{-d/p} \int_{B(x_0,r)} \left(1 + \frac{|x-x_0|}{t^{1/2}} \right)^{-M} dw \\
&\le C 2^{nd/(2p)} \left(1 + \frac{|x-x_0|}{2^{-n/2}} \right)^{-M},
\end{aligned}
$$

with $M > 0$ being large, cf. (3.18). This gives

$$
\int_{B(x_0,2r)^c} \Big(\sup_{0<t\le 2^{-n}} |T_t a(x)| \Big)^p\, dx \le C \int 2^{nd/2} \left(1 + \frac{|x-x_0|}{2^{-n/2}} \right)^{-Mp} dx \le C. \tag{6.6}
$$

It remains to estimate $\int_{B(x_0,2r)^c} (\sup_{t>2^{-n}} |T_t a(x)|)^p\, dx$.

Let P_m^t be the operator defined by

$$P_m^t f(x) = \int |f(w)| t^{-d/2} \chi_{B(0,m)}(t^{-1/2}(w-x)) \times \Big(\prod_{\beta \le \alpha} \chi_{[-m,m]}\big(t^{(|\beta|+2)/2} D^\beta V(x)\big)\Big)\, dw.$$

As a consequence of (3.18), we obtain

$$|T_t a(x)| \le \sum_{m \ge 2} b_m P_m^t a(x), \tag{6.7}$$

where $b_m \le C_q (1+m)^{-q}$ for every $q > 0$. We shall use the following lemma.

Lemma 6.8. *There is a constant $C_1 \ge 1$ independent of n such that for every bounded function f such that* $\operatorname{supp} f \subset B(x_n, 2^{1-n/2})$, *$x_n \in \mathcal{B}_n$, and every $m \ge 2$, we have*

$$P_m^t f \equiv 0 \quad \text{for } t > m^{C_1} 2^{-n}. \tag{6.9}$$

Proof. Let f be a bounded function such that $\operatorname{supp} f \subset B(x_n, 2^{1-n/2})$, $x_n \in \mathcal{B}_n$. Then, by Lemma 2.1, $\operatorname{supp} f \subset \bigcup_{k=n-C_2}^{n+C_2} \mathcal{B}_k$. Assume that $P_m^t f \not\equiv 0$. Then there is $x \in \mathbf{R}^d$ and $w \in \operatorname{supp} f$ such that

$$|t^{(|\beta|+2)/2} D^\beta V(x)| \le m \quad \text{for all } \beta,\ \beta \le \alpha, \text{ and } |t^{-1/2}(w-x)| \le m.$$

Since $D^\gamma V(w) = \sum_{\beta \le \alpha} (1/\beta!) D^{\beta+\gamma} V(x)(w-x)^\beta$ for every $\gamma \le \alpha$,

$$|D^\gamma V(w)| \le C m^{|\alpha|+1}\, t^{-(|\gamma|+2)/2}.$$

On the other hand, there is γ such that

$$|D^\gamma V(w)|^{1/(|\gamma|+2)} \ge \frac{1}{2D} 2^{(n-C_2)/2}.$$

Thus

$$2^{(n-C_2)(|\gamma|+2))/2} \le C m^{(|\alpha|+1)} t^{-(|\gamma|+2)/2}.$$

This implies $t \le m^{C_1} 2^{-n}$, which completes the proof of Lemma 6.8. □

By Lemma 6.8

$$\int_{B(x_0,2r)^c} \Big(\sup_{t>2^{-n}} |T_t a(x)| \Big)^p dx \le \Big\| \sup_{t>2^{-n}} \Big| \sum_{m=2}^{\infty} b_m P_m^t a(x) \Big| \Big\|_{L^p}^p$$

$$\le \Big\| \sum_{m=2}^{\infty} b_m \sup_{2^{-n}<t\le 2^{-n}m^{C_1}} |P_m^t a(x)| \Big\|_{L^p(dx)}^p .$$

Note that $\sup_{2^{-n}<t\le 2^{-n}m^{C_1}} |P_m^t a(x)| \le C 2^{nd/(2p)} \chi_{B(x_0,2^{-n/2}m^{C_3})}(x)$. Therefore

$$\int_{B(x_0,2r)^c} \Big(\sup_{t>2^{-n}} |T_t a(x)| \Big)^p dx \le C \sum_{m=2}^{\infty} \int b_m^p 2^{nd/2} \chi_{B(x_0,2^{-n/2}m^{C_3})}(x)\, dx \le C.$$

This ends the proof of (6.4). □

7. Local maximal functions. For a function $\varphi \in \mathcal{S}([0,\infty))$ a real number $N > 0$, and for a nonnegative integer n we define the local maximal functions $M_{\varphi,N}^{*(n)}$ and $\tilde{M}_{\varphi}^{(n)}$ by

$$M_{\varphi,N}^{*(n)} f(x) = \sup_{0<t\le 2^{-n},\ w\in\mathbf{R}^d} \{ |\varphi(tA) f(w)| (1 + t^{-1/2}|w-x|)^{-N} \}, \quad \text{and} \tag{7.1}$$

$$\tilde{M}_{\varphi}^{(n)} f(x) = \sup_{0<t\le 2^{-n}} \Big| \int_{\mathbf{R}^d} \tilde{\Phi}_t(w-x,0) f(w)\, dw \Big|, \tag{7.2}$$

where $\tilde{\Phi}_t(x,0) = t^{-d/2}\tilde{\Phi}(x/t^{1/2},\, 0)$, and $\tilde{\Phi}(x,0)$ is defined by (3.15) and (3.16), here $(x,0) \in \mathbf{R}^d \times \mathbf{R}^D$.

Let us observe, cf. [D2], that the function $x \mapsto \tilde{\Phi}(-x,0)$ is the convolution kernel of the operator $\varphi(-\Delta)$, where Δ is the Laplacian on $\mathbf{R}^d$. Therefore $\tilde{\Phi}(-x,0)$ (as a function of x) belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^d)$ and $\int \Phi(-x,0)\, dx = \varphi(0)$. Our aim in this section is to prove the following result:

Theorem 7.3. *For every l, $N > 0$ and every $p \in (0,1]$ there exists $m > 0$ such that if $\varphi \in \mathcal{S}_0^m([0,\infty))$, $\varphi(0) = 1$, then there is a constant $C > 0$ such that*

$$\|\tilde{M}_{\varphi}^{(n)}(\psi_{(n,k)} f)\|_{L^p} \le C \|(1 + 2^{n/2}|x - x_{(n,k)}|)^{-l} M_{\varphi,N}^{*(n)} f(x))\|_{L^p(dx)},$$

see Section 2 for the definitions of $\psi_{(n,k)}$ and $x_{(n,k)}$.

For $\varphi \in \mathcal{S}_0^m([0,\infty))$, $\varphi(0) = 1$ let $\psi^{(0)} \in \mathcal{S}_0^m([0,\infty))$, and $\psi \in C_c^\infty((0,\infty))$ be such that $\psi^{(0)}(\lambda) = 0$ for $\lambda > 1$ and

$$1 = \varphi(\lambda)\psi^{(0)}(\lambda) + \sum_{j=1}^{\infty} \varphi(2^{-j}\lambda)\psi(2^{-j}\lambda) \quad \text{for } \lambda \geq 0. \tag{7.4}$$

Proposition 7.5. *For every M, $\ell > 0$ there is a constant $C > 0$ such that for every $0 < t \leq 2^{-n}$ we have*

$$\left| \int \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\Psi_{2^{-j}t}(w,u)\,dw \right| \leq C2^{-jM}t^{-d/2}(1+2^{j/2}t^{-1/2}|x-u|)^{-\ell}.$$

Proof. For $\gamma \in \mathbf{N}^D$ we shall denote by D_ξ^γ the differential operator $\partial^{|\gamma|}/\partial\xi^\gamma$. From (3.15′) it follows that

$$\begin{aligned}
&\left| \int \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\Psi_{2^{-j}t}(w,u)\,dw \right| \\
&\leq \left| \int \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0) \right. \\
&\qquad \times \left. \Big[\Psi_{2^{-j}t}(w,u) - \sum_{|\gamma|<M'} \frac{1}{\gamma!} 2^{jd/2}t^{-d/2} D_\xi^\gamma \tilde{\Psi}\left(\frac{u-w}{(2^{-j}t)^{1/2}},\, 0\right)\xi^\gamma\Big]\,dw \right| \\
&\qquad + \left| \int \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0) \sum_{|\gamma|<M'} \frac{1}{\gamma!} 2^{jd/2}t^{-d/2} D_\xi^\gamma \tilde{\Psi}\left(\frac{u-w}{(2^{-j}t)^{1/2}}, 0\right)\xi^\gamma\,dw \right| \\
&= \mathrm{I}_1 + \mathrm{I}_2,
\end{aligned}$$

where $\xi = (2^{-j}tV(w), \ldots, (2^{-j}t)^{(|\beta|+2)/2}D^\beta V(w), \ldots)$.

In order to estimate I_1 we observe that $\|\xi\| \leq C2^{-j}$, whenever $w \in B_{(n,k)}$ and $0 < t \leq 2^{-n}$. By virtue of the Taylor formula and Corollary 3.14, we obtain

$$\begin{aligned}
\mathrm{I}_1 &\leq C \int_{B_{(n,k)}} 2^{-jM'}t^{-d/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{-b} 2^{jd/2}t^{-d/2}\left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} dw \\
&\leq C \int_{B_{(n,k)}} 2^{-jM'}t^{-d/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{-b} 2^{jd/2}t^{-d/2}\left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} \\
&\qquad \times \left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{\ell}\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} dw \quad \leq
\end{aligned}$$

$$\le C2^{-jM'}t^{-d/2}\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell}\int_{B_{(n,k)}}2^{jd/2}t^{-d/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{-b}$$
$$\times\left(1+\frac{|x-w|}{(2^{-j}t)^{1/2}}\right)^{\ell}dw$$

$$\le C2^{-jM'}t^{-d/2}\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell}\int_{B_{(n,k)}}2^{jd/2}t^{-d/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{-b}$$
$$\times\,2^{j\ell/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{\ell}dw$$

$$\le C2^{-jM'}t^{-d/2}2^{j(\ell+d)/2}\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell}.$$

To estimate I_2 we study each summand separately. Let

$$H(x,w)=\frac{1}{\gamma!}\psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\xi^{\gamma},$$

where $\xi=(2^{-j}tV(w),\dots,(2^{-j}t)^{(|\beta|+2)/2}D^{\beta}V(w)\dots)$. We denote by $D_w^{\kappa}H(x,w)$ the differentiation of the function $H(x,w)$ with respect to the second variable. For fixed x and u we consider Taylor expansion of the function $w\mapsto H(x,w)$ at the point u. By Corollary 3.20,

$$I_{2,\gamma}=\left|\int\frac{1}{\gamma!}\psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)2^{jd/2}t^{-d/2}D_{\xi}^{\gamma}\tilde{\Psi}\left(\frac{u-w}{(2^{-j}t)^{1/2}},0\right)\xi^{\gamma}\,dw\right|$$
$$=\left|\int H(x,w)2^{jd/2}t^{-d/2}D_{\xi}^{\gamma}\tilde{\Psi}\left(\frac{u-w}{(2^{-j}t)^{1/2}},0\right)dw\right|$$
$$=\left|\int\Big[H(x,w)-\sum_{|\kappa|<M''}\frac{1}{\kappa!}D_w^{\kappa}H(x,u)(w-u)^{\kappa}\Big]\right.$$
$$\left.\times\;2^{jd/2}t^{-d/2}D_{\xi}^{\gamma}\tilde{\Psi}\left(\frac{u-w}{(2^{-j}t)^{1/2}},0\right)dw\right|$$
$$\le\left|\int_{|w-u|\le|x-u|/2}\right|+\left|\int_{|w-u|\ge|x-u|/2}\right|$$
$$=I'_{2,\gamma}+I''_{2,\gamma}.$$

Since $0 < t \le 2^{-n}$ and M'' is large, using Taylor's formula, (2.6), and Corollary 3.14, we have

$$I'_{2,\gamma} \le C \int_{|w-u|\le|x-u|/2} |w-u|^{M''} t^{-d/2} t^{-M''/2} \left(1+\frac{|x-u|}{t^{1/2}}\right)^{-\ell} \times (2^{-j}t)^{-d/2} \left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-b} dw$$

$$\le C 2^{-jM''/2} t^{-d/2} \left(1+\frac{|x-u|}{t^{1/2}}\right)^{-\ell}$$

$$\le C 2^{-jM} t^{-d/2} \left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell}.$$

Likewise,

$$I''_{2,\gamma} \le C \int_{|w-u|\ge|x-u|/2} |w-u|^{M''} t^{-d/2} t^{-M''/2} (2^{-j}t)^{-d/2} \left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-b} dw$$

$$\le C 2^{-jM''/2} t^{-d/2} \left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell},$$

which completes the proof of Proposition 7.5. □

Corollary 7.6. *For every $M, l > 0$ there is a constant C such that for every $0 < t \le 2^{-n}$ we have*

$$\left|\int \psi_{(n,k)}(w) \tilde{\Phi}_t(w-x,0) \Psi_{2^{-j}t}(w,u)\, dw\right| \le C 2^{-jM} t^{-d/2} \left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} (1+2^{n/2}|x-x_{(n,k)}|)^{-\ell}. \tag{7.7}$$

Proof. Corollary 3.14 implies

$$\left|\int \psi_{(n,k)}(w) \tilde{\Phi}_t(w-x,0) \Psi_{2^{-j}t}(w,u)\, dw\right| \tag{7.8}$$

$$\le C \int \psi_{(n,k)}(w) t^{-d/2} \left(1+\frac{|x-w|}{t^{1/2}}\right)^{-\ell} 2^{dj/2} t^{-d/2} \left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} dw$$

$$\le C t^{-d/2} (1+2^{n/2}|x-x_{(n,k)}|)^{-\ell} \int 2^{dj/2} t^{-d/2} \left(1+\frac{|w-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell} dw \quad \le$$

$$\le Ct^{-d/2}(1+2^{n/2}|x-x_{(n,k)}|)^{-\ell}.$$

Now (7.7) follows from (7.8) and Proposition 7.5. □

Proof of Theorem 7.3. Let $0 < t \le 2^{-n}$. From (7.4) we conclude that

$$\begin{aligned}
&\left|\int \psi_{(n,k)}(w)f(w)\tilde{\Phi}_t(w-x,0)\,dw\right| \\
&= \left|\int \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\Big[\psi^{(0)}(tA)\varphi(tA)f(w)\right. \\
&\qquad \left. + \sum_{j=1}^{\infty}\psi(2^{-j}tA)\varphi(2^{-j}tA)f(w)\Big]\,dw\right| \\
&\le \left|\iint \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\Psi_t^{(0)}(w,u)\left(1+\frac{|x-u|}{t^{1/2}}\right)^N\right. \\
&\qquad\qquad \left. \times \left(1+\frac{|x-u|}{t^{1/2}}\right)^{-N}\varphi(tA)f(u)\,du\,dw\right| \\
&\qquad + \left|\sum_{j=1}^{\infty}\iint \psi_{(n,k)}(w)\tilde{\Phi}_t(w-x,0)\Psi_{2^{-j}t}(w,u)\left(1+\frac{|x-u|}{2^{-j/2}t^{1/2}}\right)^N\right. \\
&\qquad\qquad \left. \times \left(1+\frac{|x-u|}{2^{-j/2}t^{1/2}}\right)^{-N}\varphi(2^{-j}tA)f(u)\,du\,dw\right|.
\end{aligned}$$

Applying Corollaries 7.6 and 3.14 with m sufficiently large, we get

$$\begin{aligned}
&\left|\int \psi_{(n,k)}(w)f(w)\tilde{\Phi}_t(w-x,0)\,dw\right| \\
&\le C\iint \psi_{(n,k)}(w)t^{-d/2}\left(1+\frac{|x-w|}{t^{1/2}}\right)^{-\ell-N}t^{-d/2}\left(1+\frac{|w-u|}{t^{1/2}}\right)^{-\ell-N} \\
&\qquad \times \left(1+\frac{|x-u|}{t^{1/2}}\right)^N M_{\varphi,N}^{*(n)}f(x)\,dw\,du \\
&\qquad + C\sum_{j=1}^{\infty}2^{-jM}t^{-d/2}(1+2^{n/2}|x-x_{(n,k)}|)^{-\ell} \\
&\qquad \times \int\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^{-\ell-N}\left(1+\frac{|x-u|}{(2^{-j}t)^{1/2}}\right)^N M_{\varphi,N}^{*(n)}f(x)\,du \quad \le
\end{aligned}$$

$\leq C(1+2^{n/2}|x-x_{(n,k)}|)^{-\ell} M^{*(n)}_{\varphi,N} f(x)$. □

8. Proof of (1.15). Atomic decomposition. Before we turn to our proof of atomic decomposition we state some results from the theory of local Hardy spaces, cf. [Go].

Let $\tilde{\Phi}$ be a function from the Schwartz class on $\mathbf{R}^d$ such that $\int \tilde{\Phi} = 1$. For every nonnegative integer n we define the local maximal function $M^{(n)}_{\tilde{\Phi}}$ by

$$M^{(n)}_{\tilde{\Phi}} f(x) = \sup_{0<t\leq 2^{-n}} |f * \tilde{\Phi}_t(x)|, \tag{8.1}$$

where $\tilde{\Phi}_t(x) = t^{-d/2}\tilde{\Phi}(x/t^{1/2})$.

We say that f is in the local Hardy space $\mathbf{h}^p_n$ if

$$\|f\|^p_{\mathbf{h}^p_n} = \|M^{(n)}_{\tilde{\Phi}} f\|^p_{L^p} < \infty. \tag{8.2}$$

A function $\tilde{a}$ is an atom for the local Hardy space $\mathbf{h}^p_n$ associated to a ball $B(x_0,r)$, $r \leq 2^{-n/2}$ if

$$\operatorname{supp}\tilde{a} \subset B(x_0,r), \quad r \leq 2^{-n/2}, \tag{8.3}$$

$$\|\tilde{a}\|_{L^\infty} \leq (\operatorname{vol} B(x_0,r))^{-1/p}, \tag{8.4}$$

$$\text{if } r \leq 2^{-1-n/2}, \quad \text{then } \int \tilde{a}(x)x^\beta\,dx = 0 \text{ for } |\beta| \leq d\left(\frac{1}{p}-1\right). \tag{8.5}$$

The atomic quasi-norm in $\mathbf{h}^p_n$ is defined by

$$\|f\|^p_{\mathbf{h}^p_{\tilde{a},n}} = \inf\Big\{\sum_j |c_j|^p\Big\}, \tag{8.6}$$

where the infimum is taken over all decompositions $f = \sum c_j\tilde{a}_j$, where $\tilde{a}_j$ are $\mathbf{h}^p_n$ atoms and c_j are scalars.

Theorem 8.7 [Go]. *The quasi-norms $\|\ \|_{\mathbf{h}^p_n}$ and $\|\ \|_{\mathbf{h}^p_{\tilde{a},n}}$ are equivalent with constants independent of $n \in \mathbf{Z}$, that is, there exists a constant $C_p > 0$ such that for every n we have*

$$C_p^{-1}\|f\|_{\mathbf{h}^p_n} \leq \|f\|_{\mathbf{h}^p_{\tilde{a},n}} \leq C_p\|f\|_{\mathbf{h}^p_n}.$$

Moreover, if $f \in \mathbf{h}^p_n$, $\operatorname{supp} f \subset B(x, 2^{1-n/2})$, *then there are* $\mathbf{h}^p_n$ *atoms* $\tilde{a}_j$ *such that* $\operatorname{supp} \tilde{a}_j \in B(x, 2^{2-n/2})$ *and*

$$f = \sum_j c_j \tilde{a}_j, \qquad \sum_j |c_j|^p \le C\|f\|^p_{\mathbf{h}^p_n} \tag{8.8}$$

with a constant C *independent of* n.

Proof of (1.15). We first prove that for every $p \in (0,1]$ there exists $m > 0$ such that (1.15) holds for every function φ such that $\varphi \in \mathcal{S}^m_0([0,\infty))$, $\varphi(0) = 1$.

Indeed, it follows from Theorems 7.3, 8.7, Lemma 2.1, and the definition of H^p_A atoms that

$$\psi_{(n,k)} f = \sum_j c_{j,(n,k)} a_{j,(n,k)},$$

where $a_{j,(n,k)}$ are H^p_A atoms and

$$\sum_j |c_{j,(n,k)}|^p \le C\|(1 + 2^{n/2}|x - x_{(n,k)}|)^{-\ell} M^{*(n)}_{\varphi,N} f(x)\|^p_{L^p(dx)},$$

provided $\varphi \in \mathcal{S}^m_0([0,\infty))$ (with m large enough) and $\varphi(0) = 1$. Therefore

$$f = \sum_{(n,k)} \psi_{(n,k)} f = \sum_{(n,k)} \sum_j c_{j,(n,k)} a_{j,(n,k)}$$

and

$$\sum_{(n,k)} \sum_j |c_{j,(n,k)}|^p \le C \sum_{(n,k)} \|(1 + 2^{n/2}|x - x_{(n,k)}|)^{-\ell} M^*_{\varphi,N} f(x)\|^p_{L^p(dx)}.$$

Since $\sum_{(n,k)} (1 + 2^{n/2}|x - x_{(n,k)}|)^{-lp}$ is a bounded function for sufficiently large l (cf. Lemmas 2.1 and 2.2), we have

$$\sum_{(n,k)} \sum_j |c_{j,(n,k)}|^p \le C\|M^*_{\varphi,N} f\|^p_{L^p},$$

which, by (4.15), gives

$$\sum_{(n,k)} \sum_j |c_{j,(n,k)}|^p \le C\|M_\varphi f\|^p_{L^p}.$$

Now let $\varphi \in \mathcal{S}([0,\infty))$ be such that $\varphi(0) \neq 0$. Then there are constants $c_1, c_2, ..., c_{m+1}$ such that $\varphi_{(1)}(\lambda) = \sum_{n=1}^{m+1} c_n \varphi(n\lambda) \in \mathcal{S}_0^m([0,\infty))$ and $\varphi_{(1)}(0) = 1$. By the above

$$\|f\|^p_{H^p_A \text{ atom}} \leq C\|M_{\varphi_{(1)}} f\|^p_{L^p} \leq C_1 \|M_\varphi f\|^p_{L^p},$$

which completes the proof of (1.15). □

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Received: July 3rd, 1997; revised: November 2nd, 1997.