(*The pdfs in this lecture are only review material; new material was on the board, with these pdfs as background references as needed.*)

Orthogonal Decomposition Theorem

W a subspace of $\mathbb{R}^n$. Each $\boldsymbol{y} \in \mathbb{R}^n$ can be written in a unique way as

$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{z} \text{ where } \begin{cases} \widehat{\boldsymbol{y}} & \text{is in } W \\ \boldsymbol{z} & \text{is in } W^{\perp} \end{cases}$$

If $W = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_p}\}$ is an orthogonal basis for W, then formulas are

$$\begin{aligned} \widehat{\boldsymbol{y}} &= \tfrac{\boldsymbol{y}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \ldots + \tfrac{\boldsymbol{y}\cdot\boldsymbol{u_p}}{\boldsymbol{u_p}\cdot\boldsymbol{u_p}}\boldsymbol{u_p} \\ &\updownarrow \qquad\qquad\qquad\qquad \updownarrow \\ &= \text{proj}_{\boldsymbol{u_1}}\boldsymbol{y} + \ldots + \text{proj}_{\boldsymbol{u_p}}\boldsymbol{y} \\ &= \text{proj}_{W}\boldsymbol{y} \end{aligned}$$

$\widehat{\boldsymbol{y}}$ is called the orthogonal projection of y

and $\boldsymbol{z} = \boldsymbol{y} - \widehat{\boldsymbol{y}}$

$\boldsymbol{z}$ is called the component of $\boldsymbol{y}$ orthogonal to W

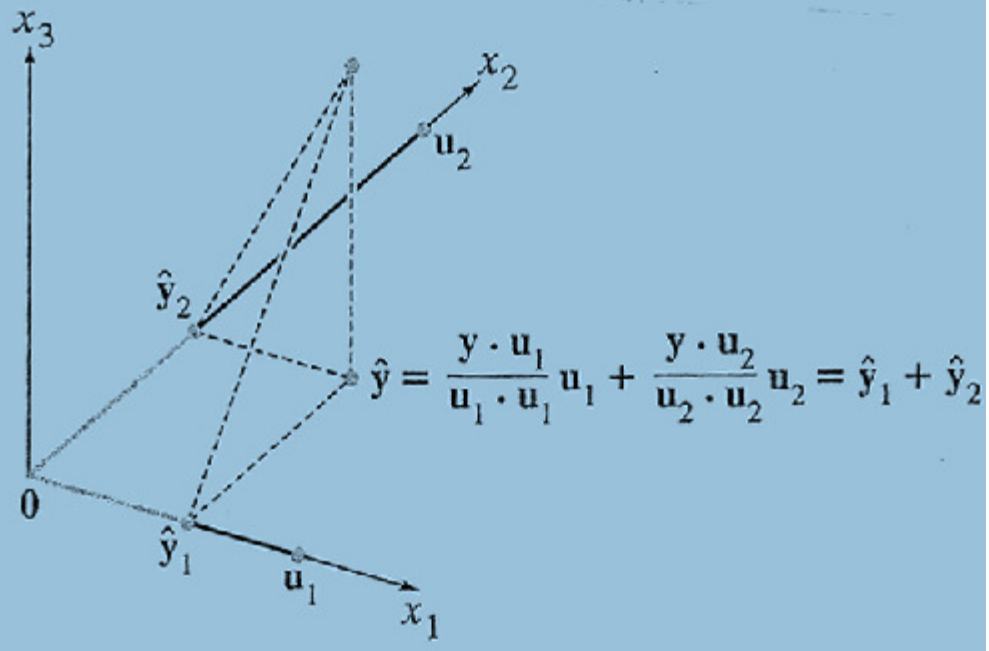


FIGURE 3 The orthogonal projection of **y** is the sum of its projections onto one-dimensional subspaces that are mutually orthogonal.

$$\begin{aligned}\widehat{\boldsymbol{y}} &= \frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{y}\cdot\boldsymbol{u}_p}{\boldsymbol{u}_p\cdot\boldsymbol{u}_p}\boldsymbol{u}_2 \\ &\quad\updownarrow \qquad\qquad \updownarrow \\ &= \text{proj}_{\boldsymbol{u}_1}\boldsymbol{y} + \text{proj}_{\boldsymbol{u}_2}\boldsymbol{y}\end{aligned}$$

Gram Schmidt Formulas

Suppose $\{\boldsymbol{x_1}, ..., \boldsymbol{x_p}\}$ is a basis for the subspace W of $\mathbb{R}^n$. Then we can convert to an orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}\}$ is an orthogonal basis for W using the formulas:

$$\boldsymbol{v_1} = \boldsymbol{x_1}$$

$$\begin{aligned}\boldsymbol{v_2} &= \boldsymbol{x_2} - \text{proj}_{\boldsymbol{v_1}}\boldsymbol{x_2} \\ &= \boldsymbol{x_2} - \frac{\boldsymbol{x_2}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1}\end{aligned}$$

$$\begin{aligned}\boldsymbol{v_3} &= \boldsymbol{x_3} - (\text{proj}_{\boldsymbol{v_1}}\boldsymbol{x_3} + \text{proj}_{\boldsymbol{v_2}}\boldsymbol{x_3}) \\ &= \boldsymbol{x_3} - \frac{\boldsymbol{x_3}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{x_3}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2}\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\boldsymbol{v_p} &= \boldsymbol{x_p} - (\text{proj}_{\boldsymbol{v_1}}\boldsymbol{x_p} + \text{proj}_{\boldsymbol{v_2}}\boldsymbol{x_p} + ... \ ... + \text{proj}_{\boldsymbol{v_{p-1}}}\boldsymbol{x_p} \\ &= \boldsymbol{x_p} - \frac{\boldsymbol{x_p}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{x_p}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2} - ... - \frac{\boldsymbol{x_p}\cdot\boldsymbol{v_{p-1}}}{\boldsymbol{v_{p-1}}\cdot\boldsymbol{v_{p-1}}}\boldsymbol{v_{p-1}}\end{aligned}$$