Definition: $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, written $\boldsymbol{u} \perp \boldsymbol{v}$,

if

$$\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$$

Note that, by definition, $\mathbf{0} \perp \boldsymbol{v}$ for every vector $\boldsymbol{v}$)

$\mathcal{B} = \{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ...\boldsymbol{u_p}\}$

Definition $\mathcal{B}$ is an orthogonal set means $\boldsymbol{u_i} \perp \boldsymbol{u_j}$ for all $i \neq j$
$\mathcal{B}$ is an orthonormal set means $\mathcal{B}$ is orthogonal and $||\boldsymbol{u_i}|| = 1$ for each i

Theorem An orthogonal set of nonzero vectors is linearly independent.

Proof is class and textbook

Therefore an orthogonal set of nonzero vectors that spans a subspace W of $\mathbb{R}^n$ is automatically an orthogonal basis for W.

If $\mathcal{B} = \{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ..., \boldsymbol{u_p}\}$ = orthogonal basis for a subspace W of $\mathbb{R}^n$
Each $\boldsymbol{y} \in W$ can be written as a linear combination of the basis vectors, and because of the orthogonality the weights (= the $\mathcal{B}$ coordinates of $\boldsymbol{y}$) are easy to find:

$$\begin{aligned} \boldsymbol{y} = \quad & c_1 \ \boldsymbol{u_1} + ... + c_i \ \boldsymbol{u_i} + ... + c_p \ \boldsymbol{u_p} \\ & \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\ = \quad & \frac{\boldsymbol{y} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}} \boldsymbol{u_1} + ... + \frac{\boldsymbol{y} \cdot \boldsymbol{u_i}}{\boldsymbol{u_i} \cdot \boldsymbol{u_i}} \boldsymbol{u_i} + ... + \frac{\boldsymbol{y} \cdot \boldsymbol{u_p}}{\boldsymbol{u_p} \cdot \boldsymbol{u_p}} \boldsymbol{u_p} \end{aligned} \qquad \textbf{(*)}$$

(*Proof done in class*): An even nicer formula (in some ways):

Definition An orthonormal basis for subspace W is an orthogonal basis $\{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ..., \boldsymbol{u_p}$ where each $||\boldsymbol{u_i}|| = 1$

If $\mathcal{B} = \{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ..., \boldsymbol{u_p}\}$ = orthonormal basis for\a subspace W of $\mathbb{R}^n$,
then each $\boldsymbol{u_i} \cdot \boldsymbol{u_i} = \mathbf{1}$ so equation (*) becomes

$$\boldsymbol{y} = (\boldsymbol{y} \cdot \boldsymbol{u_1}) \boldsymbol{u_1} + ... + (\boldsymbol{y} \cdot \boldsymbol{u_i}) \boldsymbol{u_i} + ... + (\boldsymbol{y} \cdot \boldsymbol{u_p}) \boldsymbol{u_p} \qquad \textbf{(**)}$$