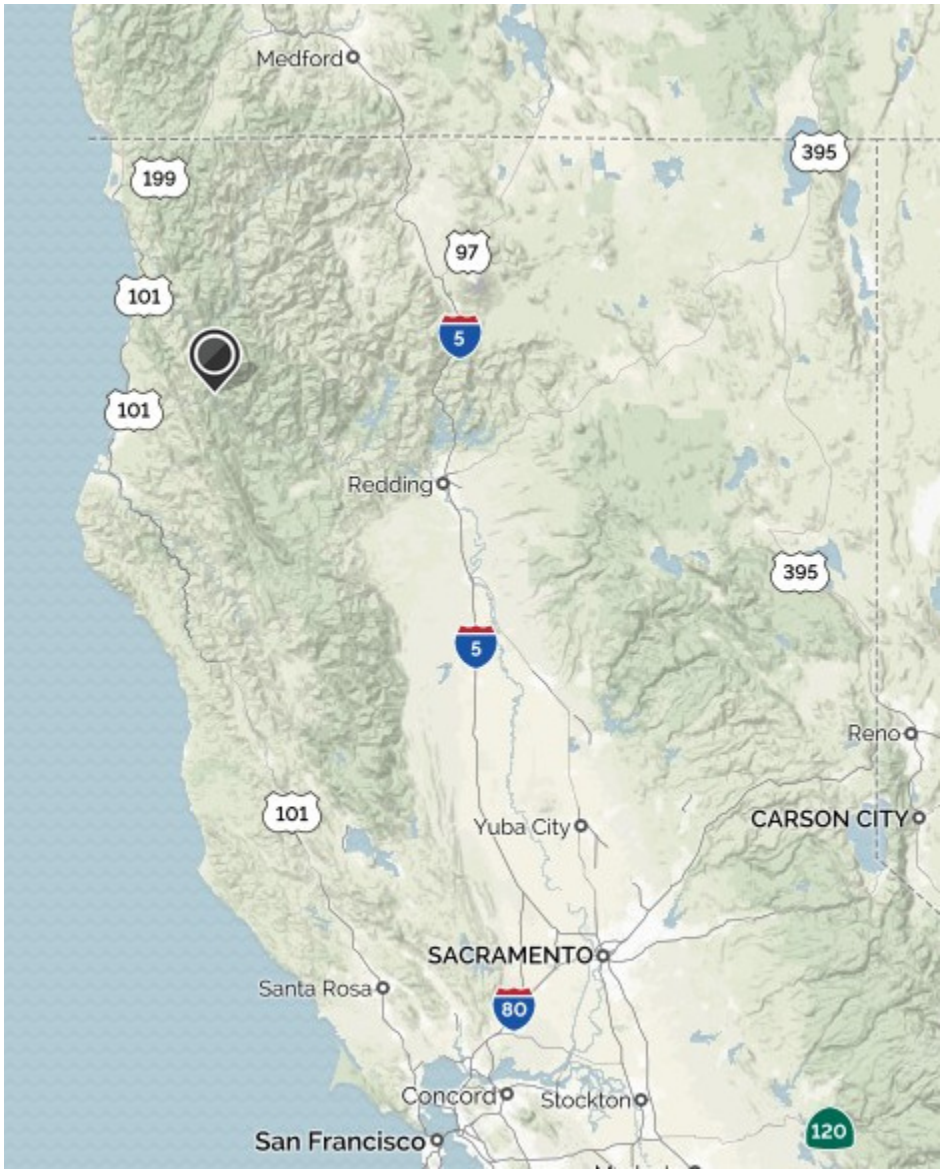

Medford
199
395
97
101
5
101
Redding
395
5
Reno
101
Yuba City
CARSON CITY
SACRAMENTO
Santa Rosa
80
Concord
Stockton
120
San Francisco

Stage Matrix

$$\begin{bmatrix} j_{k+1} \\ s_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & .33 \\ .18 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix} \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$$

From Matlab: [P,D]=eig(A)

$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$
$0.6821 - 0i$	$0.6821 + 0i$	0.3175
$-0.0624 - 0.5896i$	$-0.0624 + 0.5896i$	0.0581
$-0.0451 + 0.4256i$	$-0.0451 - 0.4256i$	0.9465

↑ Replace by

$$\begin{bmatrix} 3175 \\ 581 \\ 9465 \end{bmatrix}$$

$\lambda_1 = -0.0218 + 0.2059i$

$\lambda_2 = -0.0218 - 0.2059i$

$\lambda_3 = 0.9836$

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are <u>linearly independent</u>

If we write:

$$\begin{bmatrix} j_0 \\ s_0 \\ a_0 \end{bmatrix} = \boldsymbol{x_0} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$$

then $$\begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix} = \boldsymbol{x_n} = c_1\lambda_1^n\boldsymbol{v_1} + c_2\lambda_2^n\boldsymbol{v_2} + c_3\lambda_3^n\boldsymbol{v_3}$$

$$= c_1\lambda_1^n\boldsymbol{v_1} + c_2\lambda_2^n\boldsymbol{v_2} + c_3(0.9836)^n\boldsymbol{v_3}$$

$$|\lambda_1| = \sqrt{(-0.0218)^2 + (0.2059)^2} \approx 0.2071 < 1$$
$$|\lambda_2| = \sqrt{(-0.0218)^2 + (-0.2059)^2} \approx 0.2071 < 1$$

so

λ_1^n, λ_2^n and $(0.9836)^n$ all $\to 0$ as $n \to \infty$

Therefore $\boldsymbol{x_n} \to \mathbf{0}$ Ö !

If the stage matrix is changed to:

$$\begin{bmatrix} j_{k+1} \\ s_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & .33 \\ .50 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix} \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$$

$$\begin{array}{ccc} -0.2090 - 0.4370\,i & -0.2090 + 0.4370\,i & 2976 \\ -0.5948 + 0.4128\,i & -0.5948 - 0.4128\,i & 1421 \\ 0.4711 - 0.1384\,i & 0.4711 + 0.1384\,i & 9441 \\ \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \end{array}$$

$$\begin{array}{lll} \lambda_1 & = -0.0534 + 0.3302i & \\ \lambda_2 = \overline{\lambda_1} & = -0.0534 - 0.3302i & \lambda_3 = 1.0469 \end{array}$$

If we write:

$$\boldsymbol{x_0} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$$

then $$\boldsymbol{x_n} = c_1\lambda_1^n\boldsymbol{v_1} + c_2\lambda_2^n\boldsymbol{v_2} + c_3\lambda_3^n\boldsymbol{v_3}$$

Now, $$\boldsymbol{x_n} = c_1\lambda_1^n\boldsymbol{v_1} + c_2\lambda_2^n\boldsymbol{v_2} + c_3(1.0469)^n\boldsymbol{v_3}$$

Again, $|\lambda_1|$ and $|\lambda_2| < 1$, so $c_1\lambda_1^n\boldsymbol{v_1} + c_2\lambda_2^n\boldsymbol{v_2} \to \mathbf{0}$ as $n \to \infty$

leaving $$\boldsymbol{x_n} \approx c_3(1.0469)^n \begin{bmatrix} 2976 \\ 1421 \\ 9441 \end{bmatrix}$$

$$\boldsymbol{x_n} \approx c_3(1.0469)^n \begin{bmatrix} 2976 \\ 1421 \\ 9441 \end{bmatrix}$$

As long as the model "holds up"

- λ_3 determines the long term growth rate

  (approx 4.7% each year)

- when n is large enough so that (*)

  is a reasonable approximation, then the proportion of J, S, A owls in the population remains about the same from year to year (determined by the eigenvector $\boldsymbol{v_3}$)

  for example, juvenile proportion of population in year n

$$\frac{c_3(1.0469)^n \ (2976)}{c_3(1.0469)^n \ (2976 + 1421 + 9441)}$$

$$\approx 0.2151 \quad (21.51\ \%)$$

O_k = number of owls, R_k = number of rats (measured in "rat-units" = 1000 rats)

Model:
$$O_{k+1} = .5\,O_k + .4\,R_k$$
$$R_{k+1} = -p\,O_k + 1.1\,R_k$$

$$\boldsymbol{x_{k+1}} = \begin{bmatrix} O_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} .5 & .4 \\ -p & 1.1 \end{bmatrix} \begin{bmatrix} O_k \\ R_k \end{bmatrix} = \begin{bmatrix} .5 & .4 \\ -p & 1.1 \end{bmatrix}$$

For $p = 0.08$, A is diagonalizable with 2 distinct eigenvalues

$\lambda_1 = 1.0408$ and $\lambda_2 = .5592$

Eigenvectors $\boldsymbol{v_1} = \begin{bmatrix} .5946 \\ .8040 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} .9892 \\ .1463 \end{bmatrix}$ are linearly independent

$$\boldsymbol{x_0} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}$$
$$\boldsymbol{x_k} = c_1\lambda_1^k\boldsymbol{v_1} + c_2\lambda_2^k\boldsymbol{v_2} \approx c_1\lambda_1^k\boldsymbol{v_1}$$

↑ for large k

$$= c_1(1.0408)^k \begin{bmatrix} .5946 \\ .8040 \end{bmatrix}$$

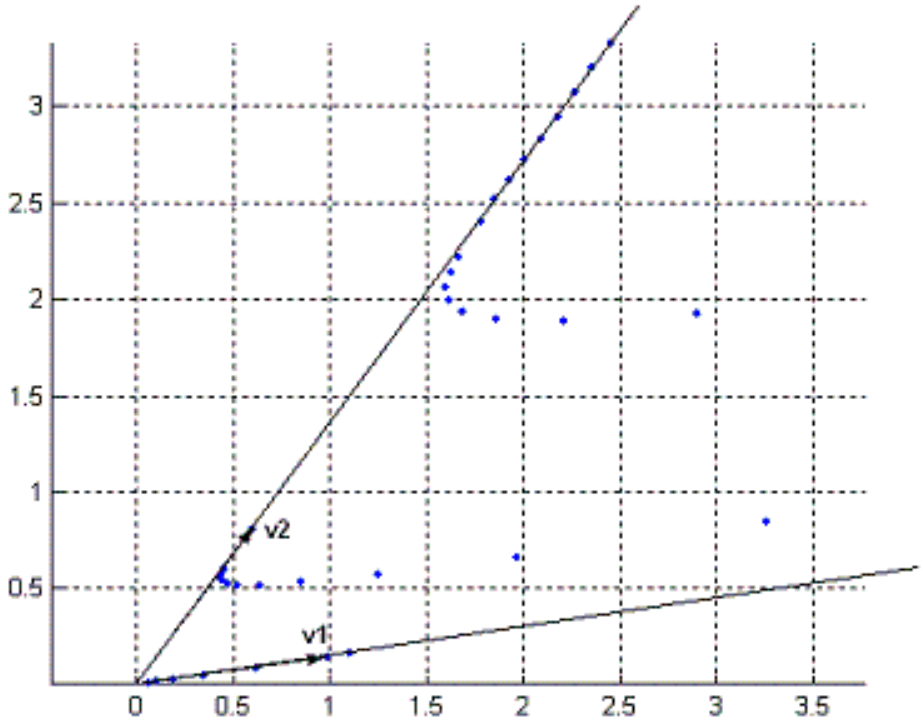
$$\boldsymbol{x_k} \approx c_1(1.0408)^k \begin{bmatrix} .5946 \\ .8040 \end{bmatrix} \quad \text{for large } k,$$

The eigenvalue λ_1 and the eigenvector $\boldsymbol{v_1}$ determine the long term behavior

- long term growth rate approx. 4.08% / yr
- ratio O_k/R_k stabilizes

  at about $\frac{.5946}{.8040} \approx \frac{3}{4}$ (3 owls/4000 rats)

Rats



Owls

Basic Properties of inner products

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$$

$$\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} = (\boldsymbol{v} + \boldsymbol{w})\boldsymbol{u}$$

$$\boldsymbol{u} \cdot \mathbf{0} = 0$$

$$(c\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$$

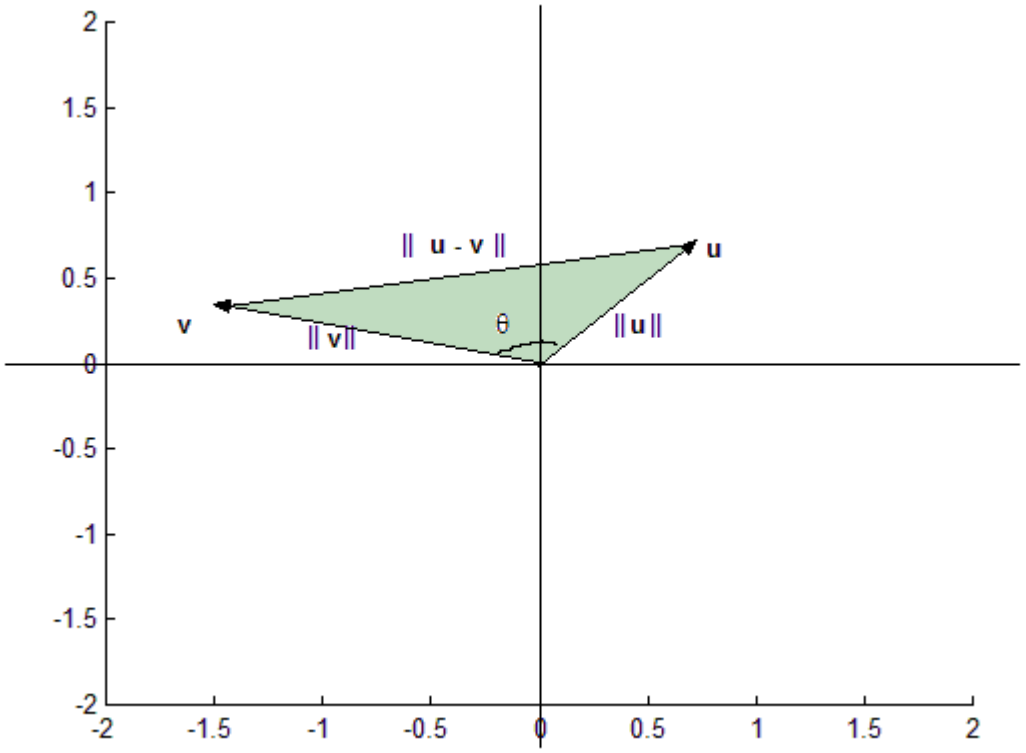
$$\boldsymbol{u} \cdot \boldsymbol{u} = u_1^2 + ... + u_n^2 \geq 0$$

$$||$$

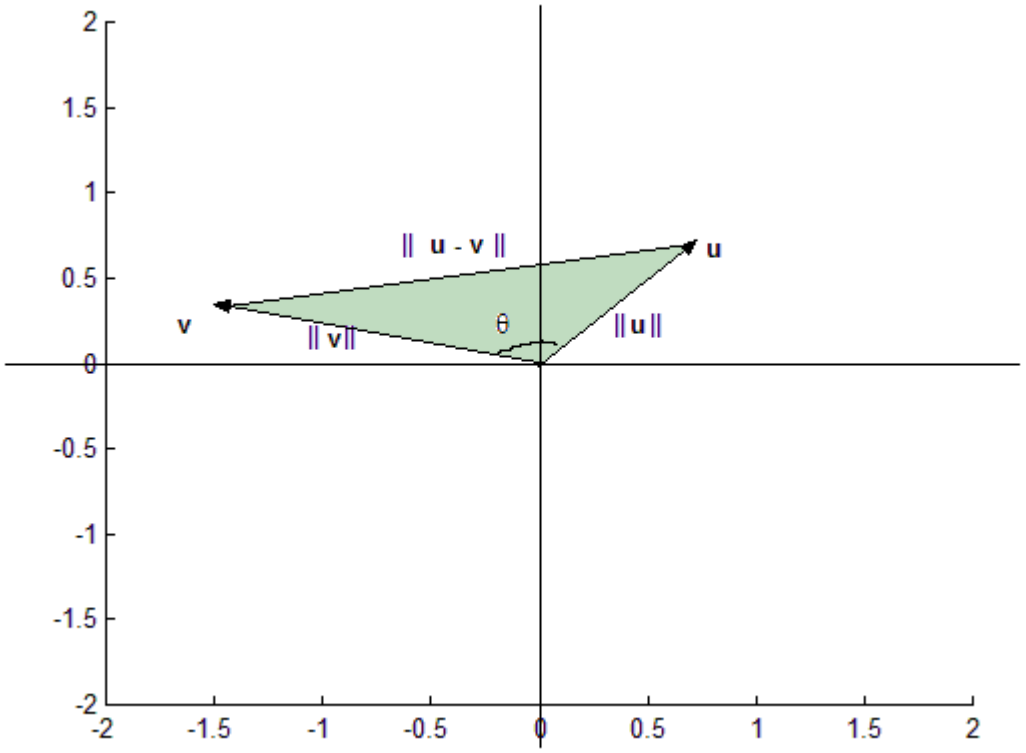
$||\boldsymbol{u}||^2$, so $\|\mathbf{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$

Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ $= \| \boldsymbol{u} - \boldsymbol{v} \|$

$$= \sqrt{(u-v)\cdot(u-v)}$$



Angle θ between two vectors
(by convention, choose $0 \leq \theta \leq \pi$)



Law of Cosines says:

$$||\boldsymbol{u}-\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$$

$$\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{u}\cdot\boldsymbol{v} = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$$

So $\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow$ if $\cos\theta = 0 \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$

Definition: $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, written $\boldsymbol{u} \perp \boldsymbol{v}$,

if

$\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$

By definition, $\mathbf{0} \perp \boldsymbol{v}$ for every vector $\boldsymbol{v}$