Review

A an $n \times n$ matrix

There is a basis $\mathcal{B}$ consisting of eigenvectors of A

$\updownarrow$

$A = PDP^{-1}$ for some diagonal matrix D

$$\left(\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\},\ A\boldsymbol{b_i} = \lambda_i \boldsymbol{b_i} \right.$$

$$\left. P = P_{\mathcal{B}} = [\boldsymbol{b_1} \ldots \boldsymbol{b_n}],\ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \right)$$

So: A is diagonalizable $\Leftrightarrow$ we can find n linearly independent eigenvectors

Theorem If $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ have different eigenvalues, then these eigenvectors are linearly independent.

so

If A has n different eigenvalues, we can pick an eigenvector for each one: $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$. These are n linearly independent eigenvectors, so A is diagonalizable.

Example from last lecture:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is lower triangular, so the eigenvalues are $\lambda = 1, 3, 6$ and therefore,

with no calculations, A must be diagonalizable.

To actually do the diagonalization, we find eigenvectors:

$\lambda = 1$ Basis for eigenspace $\left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}. \right\}$

$\lambda = 3$ Basis for eigenspace $\left\{ \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \right\}$

$\lambda = 6$ Basis for eigenspace $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Thee eigenvectors are linearly independent (they must be, because they have different eigenvalues) , so we write

$$A = PDP^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ -5 & -3 & 0 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ -5 & -3 & 0 \\ 1 & 5 & 1 \end{bmatrix}^{-1}$$

A "Better" Theorem:

Suppose

i) $\{\boldsymbol{v_1}, ..., \boldsymbol{v_p}\}$ is a set of linearly independent eigenvectors (with eigenvalues λ_1, ..., λ_p, *possibly not all different*), and

ii) $\{\boldsymbol{u_1}, ..., \boldsymbol{u_r}\}$ is a set of linearly independent eigenvectors all with the same eigenvalue λ that is different from all of λ_1, ..., λ_p.

Then $\{\boldsymbol{v_1}, ..., \boldsymbol{v_p}\} \cup \{\boldsymbol{u_1}, ..., \boldsymbol{u_r}\}$

$= \{\boldsymbol{v_1}, ..., \boldsymbol{v_p}, \boldsymbol{u_1}, ..., \boldsymbol{u_r}\}$ is linearly independent

Proof: Done in class; see notes posted online in "Supplementary proofs about Diagonalization."

To make use of this theorem in a diagonalization question for an $n \times n$ matrix A :

Find the eigenvalues λ_1, ..., λ_p
For each eigenspace Nul $(A - \lambda_i I)$: find a basis $\mathcal{B}_i$ and dimension of eigenspace (number of vectors in $\mathcal{B}_i$) by solving $(A - \lambda_i I)\boldsymbol{x} = \boldsymbol{0}$

Uniting the bases together (*see discussion in class on in the supplement online)*

$\mathcal{B}_1 \cup ... \cup \mathcal{B}_p$ gives a set of linearly independent eigenvectors

↑

How many?

The number of linearly independent eigenvectors in this collection is the sum of the dimensions of the eigenspaces.

So for an $n \times n$ matrix A :

A is diagonalizable $\Leftrightarrow$ sum of dimensions of the eigenspaces $= n$

Examples

1) From earlier:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ Eigenvalues $\lambda = 1, 3, 6.$

A must be diagonalizable.

To actually do it, need to find eigenvectors

$\lambda = 1$ Basis for eigenspace $\left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}. \right\}$

$\lambda = 3$ Basis for eigenspace $\left\{ \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \right\}$

$\lambda = 6$ Basis for eigenspace $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Sum of the dimensions of the eigenspaces $= 3$: A is diagonalizable

2) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$\det(A - \lambda I) = C(\lambda) = -(\lambda - 3)^2(\lambda + 1) = 0$

Eigenspace for $\lambda = 3$ (multiplicity of eigenvalue $= 2$)

is $\{\boldsymbol{x} : \boldsymbol{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, x_1, x_3 \text{ free}\}$

Eigenspace for $\lambda = -1$ (multiplicity of eigenvalue1)

$\{\boldsymbol{x} : \boldsymbol{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, x_3 \text{ free}\}$

The sum of the dimensions of the eigenspaces is 3. Therefore A is diagonalizable.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$$

3) Dimension of an eigenspace <u>can</u> be smaller than the multiplicity of the eigenvalue.

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \det(B - \lambda I) = C(\lambda) = (2-\lambda)^2(4-\lambda) = 0$$

Eigenvalues : 2 (multiplicity 2) ;
4 (multiplicity 1).

<u>Eigenspace for $\lambda = 2$</u>, solve $(A - 2I)\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{\boldsymbol{x} : \boldsymbol{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$, the <u>eigenspace has dimension</u> 1 ($<$ multiplicity 2 of eigenvalue)

<u>Eigenspace for $\lambda = 4$</u>: $\{\boldsymbol{x} : \boldsymbol{x} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\}$ <u>also has dimension 1</u>

Sum of dimensions of eigenspaces $= 1 + 1 = 2 < 3$; A is <u>not</u> diagonalizable

Theorem (no proof in class or in textbook; see the notes online "Supplementary proofs about Diagonalization")

For each eigenvalue λ_i of A :

(dimension of eigenspace) $\leq$ (multiplicity of the eigenvalue λ_i)

The last example shows that " < " can happen for an eigenvalue (in which case A is <u>not</u> diagonalizable).