A: $m \times n$ matrix, and $T : \mathbb{R}^n \to \mathbb{R}^m$ is $T(\boldsymbol{x}) = A\boldsymbol{x}$

$$\begin{array}{lcl} \dim \text{Nul } A & = & \text{number of nonpivot cols} \\ \dim \text{Col } A & = & \text{number of pivot columns} \end{array}$$

so

$$\begin{array}{ccc} \textbf{dim Col}(\boldsymbol{A}) & + \textbf{dim}(\textbf{Nul } \boldsymbol{A}) = \boldsymbol{n} \\ \| & \| \\ \textbf{dim range}(\boldsymbol{T}) & + \textbf{dim ker}(\boldsymbol{T}) = \boldsymbol{n} \\ \| \textbf{ (def)} & \| \textbf{ (def)} \\ \textbf{rank}(\boldsymbol{A}) & + \textbf{nullity}(\boldsymbol{A}) = \boldsymbol{n} \end{array}$$

Rank Theorem $\longrightarrow$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \sim \ldots \sim U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{x} \longrightarrow A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \boldsymbol{x}$$

in $\mathbb{R}^3$ — in $\mathbb{R}^2$

Rank $A = r$
= number of pivot columns in $A = 2$

Nul A: Basis $\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ — dimension 1

Col A : Basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ — dimension 2

$$\begin{array}{l} \dim \text{Nul } A + \dim \text{Col } A = 3 \\ \text{Nullity } A + \text{Rank } A = 3 \end{array}$$

In general, for an $m \times n$ matrix A:

$$\begin{array}{l} \dim \text{Nul } A + \dim \text{Col } A = n \\ (n - r) + r = n \end{array}$$

$$\text{Nullity } A + \text{Rank } A = n$$

Two vectors $\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ are

called perpendicular (or <u>orthogonal</u>) if $\boldsymbol{r}^T\boldsymbol{x} = \mathbf{0}$

$$\boldsymbol{r}^T\boldsymbol{x} = [r_1, r_2, ., r_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [r_1x_1 + r_2x_2 + ... + r_nx_n] = [0]$$

So $\boldsymbol{r} \perp \boldsymbol{x}$ if and only if $r_1x_1 + r_2x_2 + ... + r_nx = 0$

$\boldsymbol{r}^T\boldsymbol{x}$ is viewed as a scalar (though technically, in our definition, a 1×1 matrix)

$\boldsymbol{r}^T\boldsymbol{x}$ is called the <u>inner product</u> or <u>scalar product</u> or <u>dot product</u> if the vectors. We sometimes write $\boldsymbol{r}^T\boldsymbol{x}$ **as** $\boldsymbol{r} \cdot \boldsymbol{x}$.

(*We'll say more about this definition in Chapter 6, but it's a characterization of perpendicularity you probably have seene somewhere else: in Calc III or a physics course that used vectors to describe forces.*)

Nul A is perpendicular to Row A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \text{Row}_1\, A \\ \text{Row}_2\, A \\ \vdots \\ \text{Row}_n\, A \end{bmatrix}$$

If $\boldsymbol{x}$ is in Nul A, then

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \dots & \boldsymbol{a_{1n}} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \vdots \\ \boldsymbol{x_n} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{x^T \text{Row}_1 A} \\ \boldsymbol{x}^T \text{Row}_2 A \\ \vdots \\ \boldsymbol{x}^T \text{Row}_m A \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ so}$$

$\boldsymbol{x}$ is perpendicular to each vector in $\{\text{Row}_1 A, \dots, \text{Row}_m A\}$. These vectors span Row A, and it is easy to check that

$$\boldsymbol{x} \cdot \underbrace{(c_1 \text{Row}_1 A + \dots + c_n \text{Row}_n A)}_{\uparrow \text{ an arbitrary vector in Row } A} = c_1 \boldsymbol{x} \cdot \text{Row}_1 A + \dots + c_n \boldsymbol{x} \cdot \text{Row}_n A = 0$$

So we say that Nul $A \perp$ Row A

Example (continued

$$A \quad = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \sim ... \sim U \quad = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Row A = Span $\{(1\,0\,2),\ (1\,1\,4)\}$, a subspace of $\mathbb{R}^3$

Basis for Row A : $\{(1,0,2),\ \ (0,1,2)\}$ (*the nonzero rows in an echelon form of* A)

dimension Row $A = 2$
||
dimension Col A

For an $m \times n$ matrix A

$$\begin{array}{lll} \text{dim Nul } A & + \text{dim Col } A & = n \\ & || & \\ & + \text{dim Row } A & = n \\ || & || & \\ \text{Nullity } A & + \text{Rank } A & = n \\ || & || & \\ (n-r) & + \quad r & = n \end{array}$$

For $T(\boldsymbol{x}) = A\boldsymbol{x}$: *Nul A is same as kernel of T*
Col A is same as range of T

A^T *is* $n \times m$ and defines a mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$

Since the rows (olumns) of A^T are the columns (rows) of A,

Col A^T = Row A, a subspace of $\mathbb{R}^n$
Row A^T = Col A, a subspace of $\mathbb{R}^m$

Just as for any matrix, Row $A^T \perp$ Nul A^T.

Example (continued)

$$\begin{array}{ccccc} \mathbb{R}^3 & & & & \mathbb{R}^2 \\ \boldsymbol{x} & \longrightarrow & A\boldsymbol{x} & & \\ A^T\boldsymbol{y} & \longleftarrow & \boldsymbol{y} & & \end{array}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}$$

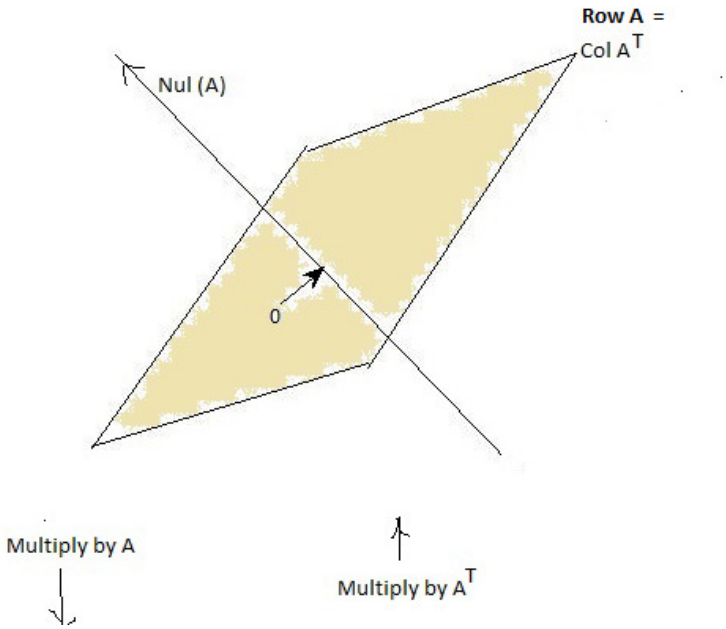
$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

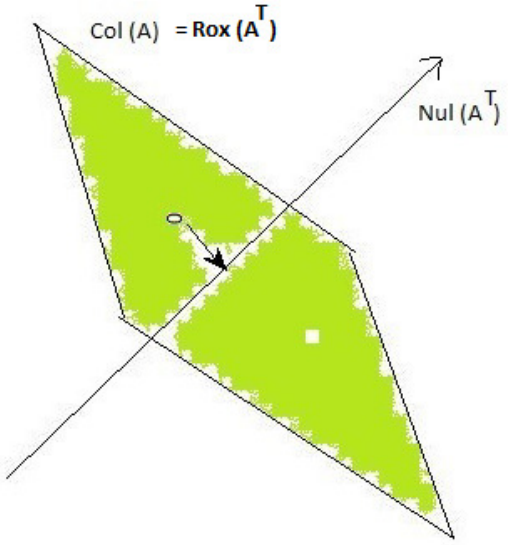
Row A^T Basis $\{(1,0),\ (0,1)\}$

$\quad ||$

Col A Basis $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

(*The two spaces are the same; but our methods for finding bases for row and columns spaces produced two (equally good) bases for this space*)


Row A =
Col A^T
Nul (A)
0
Multiply by A
Multiply by A^T


Col (A) = Rox (A^T)
Nul (A^T)
0

A schematic diagram of the four “fundemenjtal subspaces” associated with a matrix A

Col A^T is the same as Row A, and
Row A^T is the same as Col A
so they aren't really “missing” from the picture

