Practice

Suppose $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an invertible 2×2 matrix.

Then

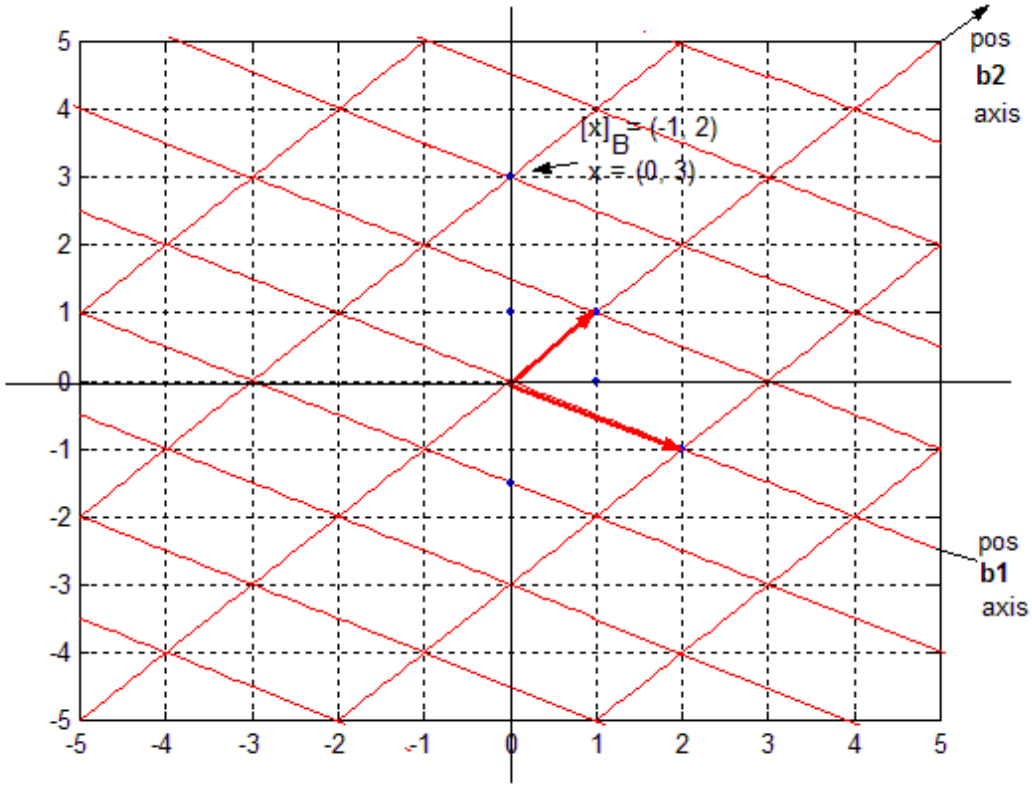
$$\det(A^{-1}) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$= \frac{ad-bc}{ad-bc} = 1$$

Is this correct ?

Extra Example on Changing Bases

Use basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$



Geometrically: where is $\boldsymbol{x}$ if $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$?

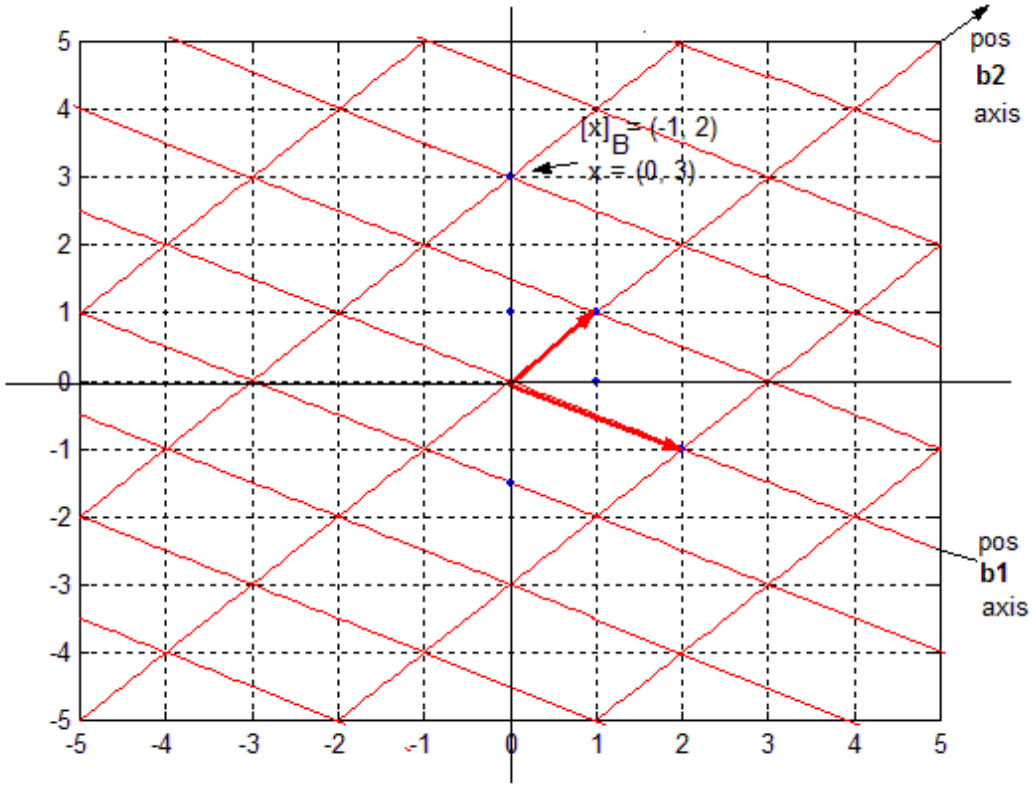
What are (approx.)the $\mathcal{B}$-coordinates for $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$?

$[\boldsymbol{b_1}]_{\mathcal{B}} = \begin{bmatrix} ** \\ ** \end{bmatrix}$ $[\boldsymbol{b_2}]_{\mathcal{B}} = \begin{bmatrix} ** \\ ** \end{bmatrix}$

Still using basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

$\boldsymbol{b_1}$ $\boldsymbol{b_2}$

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$



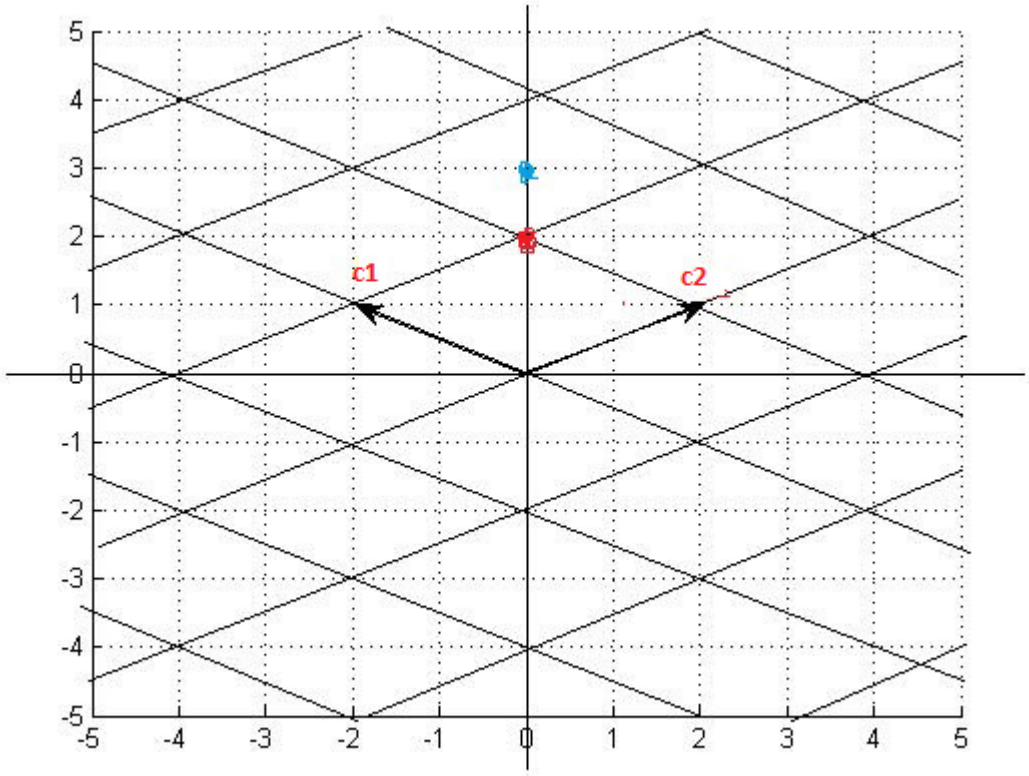
$$P_{\mathcal{B}} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ means ... ?

$$P_{\mathcal{B}}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ means ... ?

Pick another new basis, $\mathcal{C}$, for $\mathbb{R}^2$

$A = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible, so its columns are a basis

$$\mathcal{C} = \left\{ \boldsymbol{c_1},\ \boldsymbol{c_2} \right\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \quad P_{\mathcal{C}} = A$$



Meaning of :

$$P_{\mathcal{C}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad P_{\mathcal{C}} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} P_{\mathcal{B}} [\boldsymbol{x}]_{\mathcal{B}} = P_{\mathcal{C}}^{-1} \boldsymbol{x} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = [\boldsymbol{x}]_{\mathcal{C}} \quad (*)$$

↑ ↑

multiplying by $P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$ converts the $\mathcal{B}$-coordinate vector into the $\mathcal{C}$-coordinate vector

The topic of converting from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is discussed in more detail in Section 4.7, which we will not cover. The method there is a similar idea, but presented as a one-step process that is implemented by a row reduction. Generally, the method in 4.7 is more efficient. Just take note that it's there.

A: $m \times n$ matrix, and

$T : \mathbb{R}^n \to \mathbb{R}^m$ is $T(\boldsymbol{x}) = A\boldsymbol{x}$

dim Nul A = number of nonpivot cols

dom Col A = number of pivot columns

so

Rank Theorem

$$\begin{array}{ccc} \textbf{dim (Col } A\textbf{)} & + & \textbf{dim Nul}(A) = n \\ || & & || \\ \textbf{dim range}(T) & + & \textbf{dim ker}(T) = n \\ || & & || \\ \textbf{rank}(A) & + & \textbf{nullity}(A) = n \end{array}$$

Suppose $T : \mathbb{R}^5 \to \mathbb{R}^7$ is linear

the largest possible value for dim range$(T) =$
&
the corresponding value for dim ker$(T) =$
(T *cannot be onto!*)

if the range of T is a plane through $\mathbf{0}$ in $\mathbb{R}^7$
what is dim$\{\boldsymbol{x} : T(\boldsymbol{x}) = \mathbf{0}\}$?

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear

If dim ker$(T) = 1$, then dim range $(T) =$

If dim ker$(T) = 2$, then dim range $(T) =$

If dim ker$(T) = 3$, then dim range $(T) =$

If dim ker$(T) = 0$, then dim range $(T) =$
(T *cannot be one-to-one !*)

If dim $\to$ ker$(T) = \mathbf{0}$, then
$\uparrow$
IMPOSSIBLE

(T *cannot be one-to-one !*)