V vector space, with basis $\mathcal{B} = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$

The mapping $T : V \to \mathbb{R}^n$ given by $T(\boldsymbol{x}) = [\boldsymbol{x}]_\mathcal{B}$
is a linear transformation that is one-to-one and onto
$\uparrow$
an isomorphism between V and $\mathbb{R}^n$

Why: illustrate with $n = 2$ (only to save space: argument is the same for any n)

V $\xrightarrow{[\ ]_\mathcal{B}}$ $\mathbb{R}^2$

$\boldsymbol{x} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2}$ $\longrightarrow$ $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$c\boldsymbol{x} = cc_1\boldsymbol{b_1} + cc_2\boldsymbol{b_2}$ $\longrightarrow$ $\begin{bmatrix} cc_1 \\ cc_2 \end{bmatrix} = c\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$[c\boldsymbol{x}]_\mathcal{B} = c[\boldsymbol{x}]_\mathcal{B}$

$\boldsymbol{y} = d_1\boldsymbol{b_1} + d_2\boldsymbol{b_2}$ $\longrightarrow$ $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$\boldsymbol{x} + \boldsymbol{y} = (c_1 + d_1)\boldsymbol{b_1} + (c_2 + d_2)\boldsymbol{b_2}$ $\longrightarrow$ $\begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$[\boldsymbol{x} + \boldsymbol{y}]_\mathcal{B} = [\boldsymbol{x}]_\mathcal{B} + [\boldsymbol{y}]_\mathcal{B}$

$\mathbf{0} = 0\boldsymbol{b_1} + 0\boldsymbol{b_2}$ $\longrightarrow$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$

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if $\boldsymbol{x} \neq \boldsymbol{y}$ $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\boldsymbol{x}]_\mathcal{B} \neq [\boldsymbol{y}]_\mathcal{B} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

(so the mapping is one-to-one)

let $\boldsymbol{x} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2}$ $\quad$ For any $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $[\boldsymbol{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(*so the mapping is onto* $\mathbb{R}^2$)

$$c\boldsymbol{x} + d\boldsymbol{y} = \boldsymbol{z} \quad \Leftrightarrow \quad [c\boldsymbol{x} + d\boldsymbol{y}]_{\mathcal{B}} = [\boldsymbol{z}]_{\mathcal{B}}$$
$$\Leftrightarrow \quad c[\boldsymbol{x}]_{\mathcal{B}} + d[\boldsymbol{y}]_{\mathcal{B}} = [\boldsymbol{z}]_{\mathcal{B}}$$

so: linear dependence relationships in V correspond perfectly with linear dependence relation of the coordiate vectors

$$c\boldsymbol{x} + d\boldsymbol{y} = \boldsymbol{0} \quad \Leftrightarrow \quad c[\boldsymbol{x}]_{\mathcal{B}} + d[\boldsymbol{y}]_{\mathcal{B}} = [\boldsymbol{0}]_{\mathcal{B}} = \boldsymbol{0}$$

$\boldsymbol{x}, \boldsymbol{y}$ are linearly dependent or independent $\Leftrightarrow$ the same is true of the coordinate vectors (and similar for any number of vectors)

Example In $M_{3,2}$ (the vector psace of all of 3×2 matrices) is the set

$$\left\{ \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \right\}$$

linearly independent or dependent? Pick the "standard basis" for $M_{3,2}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The coordinate vectors for the three matrices are $\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

Check whether these are linearly independent (in $\mathbb{R}^6$)

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 2 & -3 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The coordinate vectors (columns) are linearly independent because every columkn is a pivot column.

Example Are $\boldsymbol{p}(t) = 1 + 2t$,
$\boldsymbol{q}(t) = -2 + 3t^2$, and
$\boldsymbol{r}(t) = 1 + 2t - t^2$

linearly independent vectors in $\mathbb{P}_2$???

If $\boldsymbol{c_1 p(t) + c_2 q(t) + c_3 r(t) = 0}$, must $\boldsymbol{c_1 = c_2 = c_3 = 0}$ **?**

Use the standard basis $\mathcal{B}\{1, t, t^2\}$ and coordinate mapping $[\ ]_{\mathcal{B}}$ to "translate" to <u>equivalent</u> question about coordinate vectors in $\mathbb{R}^3$:

If $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. must $c_1 = c_2 = c_3 = 0$?

Yes, because

$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & -1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ has 3 pivot positions.

Proved in class:

i) If V has a basis $\mathcal{B} = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$, then every set containing more than n vectors in V must be linearly dependent.

ii) If V has a basis $\mathcal{B} = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$, then every basis for V must also contain exactly n vectors.

Definition If V has a basis $\mathcal{B} = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$, we say that the dimension of V is n :

$$\dim V = n$$

Examples

$\dim \mathbb{R}^n = n$

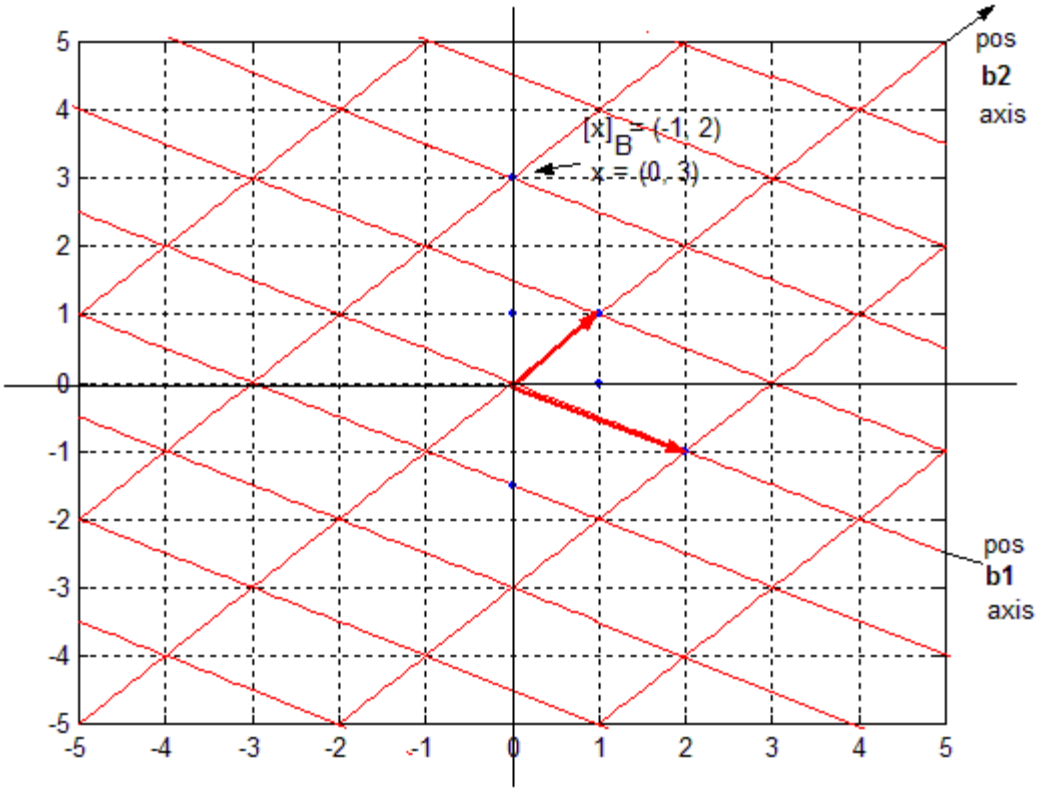
$\dim \mathbb{P}_n = n + 1$

$\dim M_{m,n,} = mn$

$\dim \{\mathbf{0}\} = 0$ (*by definition: recall that* $\{\mathbf{0}\}$ *does not have a basis*)/

dim Col A = the number of pivot columns in A.

Example in $\mathbb{R}^2$, pick basis $\mathcal{B} = \{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$, $P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$



Geometrically: where is $\boldsymbol{x}$ if $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$?

$P_{\mathcal{B}} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means ... ?

$P_{\mathcal{B}}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ means ... ?

Approx. what are the $\mathcal{B}$-coordinates for $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$?

Rotating the Coordinate System

ϕ = some fixed angle.

Rotation matrix $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is invertible

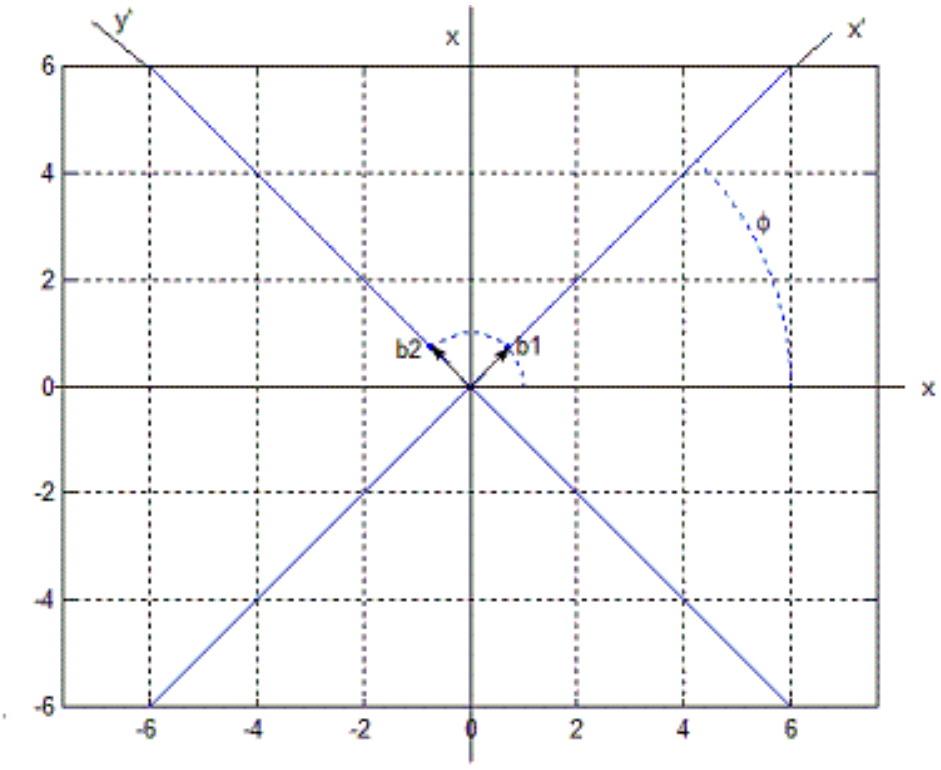
so columns $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ are linearly independent, span $\mathbb{R}^2$

$\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \{\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

$\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ determine a new set of coordinate axes

Call them the x' and y' axes: so $\mathcal{B}$-coordinates are the same thing as x'-y' coordinates:

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Converting $\begin{bmatrix} x' \\ y' \end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix}$

$\mathcal{B}$-coordinates $\leftrightarrow$ standard coordinates

$$\underset{\begin{matrix}\uparrow \\ \boldsymbol{b_1}\end{matrix}}{\overset{P_{\mathcal{B}}}{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}} \underset{\begin{matrix}\uparrow \\ \boldsymbol{b_2}\end{matrix}}{\begin{bmatrix} x' \\ y' \end{bmatrix}} = \overset{[\boldsymbol{x}]_{\mathcal{B}} = \quad \boldsymbol{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} \qquad (*)$$

so $$\begin{cases} x = x'\cos\phi - y'\sin\phi \\ y = x'\sin\phi + y'\cos\phi \end{cases}$$

$$\overset{P_{\mathcal{B}}^{-1}}{\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}} \overset{\boldsymbol{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{[\boldsymbol{x}]_{\mathcal{B}}}{\begin{bmatrix} x' \\ y' \end{bmatrix}} \qquad (**)$$

$$\begin{cases} x' = \quad x\cos\phi + y\sin\phi \\ y' = -x\sin\phi + y\cos\phi \end{cases}$$

What does the graph of $5x^2 - 4xy + 5y^2 = 63$ look like ?

Rotate axes counterclockwise by $\frac{\pi}{4}$

(*Don't worry here about why I decided to do this: it's a topic in Chapter 7)*)

Convert to x'-y' coordinates

$$\begin{cases} x = x'\cos\phi - y'\sin\phi \\ y = x'\sin\phi + y'\cos\phi \end{cases}$$

$$\begin{aligned} 63 \quad &= 5x^2 - 4xy + 5y^2 \\ &= \quad 5(x'\cos\phi - y'\sin\phi)^2 \\ &\quad\; - 4(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) \\ &\quad\; + 5(x'\sin\phi + y'\cos\phi)^2 \end{aligned}$$

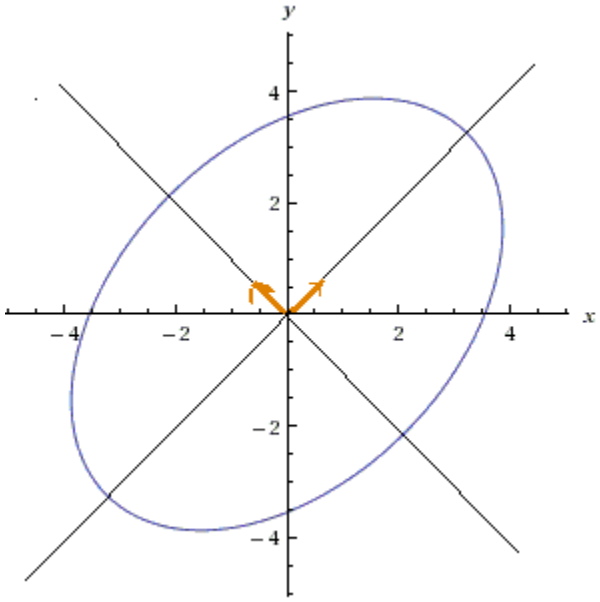
Letting $\phi = \frac{\pi}{4}$, we get

$$\begin{aligned} 63 = \; & 5(x'\tfrac{\sqrt{2}}{2} - y'\tfrac{\sqrt{2}}{2})^2 \\ & - 4(x'\tfrac{\sqrt{2}}{2} - y'\tfrac{\sqrt{2}}{2})(x'\tfrac{\sqrt{2}}{2} + y'\tfrac{\sqrt{2}}{2}) \\ & + 5(x'\tfrac{\sqrt{2}}{2} + y'\tfrac{\sqrt{2}}{2})^2 = 63 \end{aligned}$$

Simplifying gives

$$3(x')^2 + 7(y')^2 = 63, \text{ or}$$

$$\text{or } \tfrac{(x')^2}{21} + \tfrac{(y')^2}{9} = 1$$



$$\tfrac{(x')^2}{21} + \tfrac{(y')^2}{9} = 1$$

$$5x^2 - 4xy + 5y^2 = 63$$