In $\mathbb{R}^n$, inner product:

$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = u_1 v_1 + ... + u_n v_n.$
$||$
$<\boldsymbol{u}, \boldsymbol{v}>$

For any vector space: inner product on V

some rule: for $\boldsymbol{u}, \boldsymbol{v} \in V$, $< \boldsymbol{u}, \boldsymbol{v} > =$ scalar
such that

a) $< \boldsymbol{u}, \boldsymbol{v} > = < \boldsymbol{v}, \boldsymbol{u} >$
b) $< \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} > = < \boldsymbol{u}, \boldsymbol{w} > + < \boldsymbol{v}, \boldsymbol{w} >$
c) $< c\boldsymbol{u}, \boldsymbol{v} > = c < \boldsymbol{u}, \boldsymbol{v} >$
d) $< \boldsymbol{u}, \boldsymbol{u} > \geq 0$ and
$< \boldsymbol{u}, \boldsymbol{u} > = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$.

(*These are the same properties that the inner product $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$ has.*
In $\mathbb{R}^n$, we sometimes also write $\boldsymbol{u} \cdot \boldsymbol{v} = < \boldsymbol{u}, \boldsymbol{v} >$

In any vector space V with an inner product defined, can use the inner product to define

length : $|| \boldsymbol{u} || = < \boldsymbol{u}, \boldsymbol{u} >^{1/2}$

distance : $|| \boldsymbol{u} - \boldsymbol{v} || = < \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} >^{1/2}$

orthogonal: $\boldsymbol{u} \perp \boldsymbol{v}$ if anf only if $< \boldsymbol{u}, \boldsymbol{v} > = 0$

Example: Let $C[-\pi, \pi]$ continuous real-valued
functions with domain $[-\pi, \pi]$

Call this vector space C, for short

For f, g in C, define

$< f, g >$ = the number $\int_{-\pi}^{\pi} f(x) g(x)\, dx$

$< f, g >$ is an inner product on C

norm (or "length") of f: $||f|| = < f, f >^{1/2} = (\int_{-\pi}^{\pi} f^2(x)\, dx)^{1/2}$

distance between f and g: $|| f - g || = (\int_{-\pi}^{\pi} (f(x) - g(x))^2\, dx)^{1/2}$

f and g are orthogonal $\qquad < f, g > \; = \int_{-\pi}^{\pi} f(x)g(x)\,dx \; = 0$

For subspace W of C :

$$W^{\perp} = \{f : \; < f, g > \; = 0 \text{ for all } g \text{ in } W\}$$

If subspace W of C has orthogonal basis $\{g_1, ..., g_n\}$ and $f \in C$, then can write

$$f = (\text{proj}_W f) + g \text{ where } \text{proj}_W f \in W \text{ and } g \in W^{\perp}$$

$\text{proj}_W f$ and g are unique

$$(\text{proj}_W f) = \frac{<f, g_1>}{<g_1, g_1>} g_1 + ... + \frac{<f, g_n>}{<g_n, g_n>} g_n$$

$\text{proj}_W f$ is the function in W closest to f, that is the function in W for which

$$||f - (\text{proj}_W f)|| < ||f - g|| \text{ for all } \; g \in W, g \neq f$$

If $\{f_1, f_2, ..., f_n\}$ is a basis for subspace W of C, we can use same Gram Schmidt formulas as in $\mathbb{R}^n$ to convert to an orthogonal basis using the same Gram Schmidt process formulas.

For all the definitions in the preceding example, continuous functions with domain $[a, b]$ would have worked just as well. Then for course, $< f, g > \; = \int_a^b f(x)g(x)\,dx$

Example In what follows, let $C[-\pi, \pi]$ with inner product

$$< f, g > = \int_{-\pi}^{\pi} f(x)g(x)d\,dx$$

In this example, the domain $[-\pi, \pi]$ is important, although we could use any closed interval with length 2π (= one period of the sin and cos functions)

Let $W = \text{Span}\{1,\ \cos x,\ \cos 2x,\ ...,\cos nx,\ \sin x,\ \sin 2x,...,\ \sin nx\}$

Orthogonal basis! For any two functions * and ** in this collection, $\int_{-\pi}^{\pi} (*)(**)\,dx = 0$ (*see the class notes, or the written notes on Inner Product Spaces*))

A function in W looks like

$$c_0 + a_1\cos x + ... + a_n\cos nx + b_1\sin x + ... + b_n\sin nx$$ (*a trigonometric polynomial*)

The function W closest to f is

$$\text{proj}_W f = \frac{<f(x),1>}{<1,1>}1$$

$$+ \frac{<f(x),\cos x>}{<\cos x,\cos x>}\cos x + ... + \frac{<f(x),\cos nx>}{<\cos nx,\cos nx>}\cos nx$$

$$+ \frac{<f(x),\sin x>}{<\sin x,\sin x>}\sin x + ... + \frac{<f(x),\sin nx>}{<\sin nx,\sin nx>}\sin nx$$

To take a specific example, suppose $f(x) = x$:

$$\text{proj}_W f = \frac{<x,1>}{<1,1>}1$$

$$+ \frac{<x,\cos x>}{<\cos x,\cos x>}\cos x + ... + \frac{<x,\cos nx>}{<\cos nx,\cos nx>}\cos nx$$

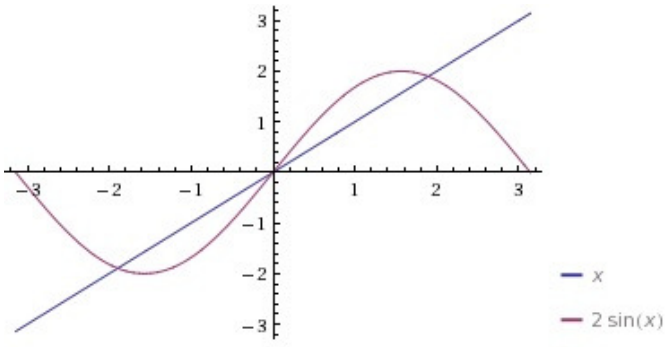
$$+ \frac{<x,\sin x>}{<\sin x,\sin x>}\sin x + ... + \frac{<x,\sin nx>}{<\sin nx,\sin nx>}\sin nx$$

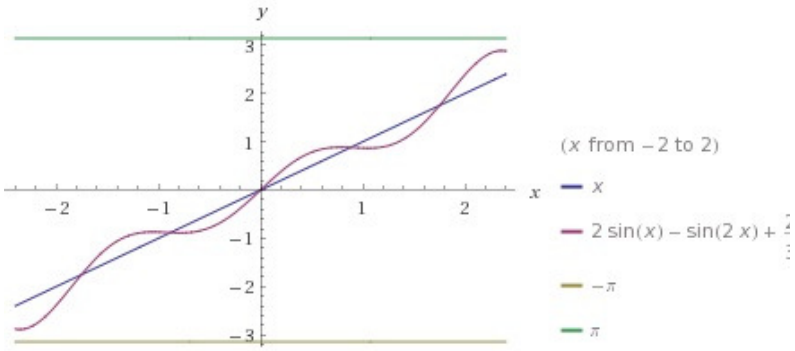
(see sample calculations in the printed notes: all the blue and red terms are 0)

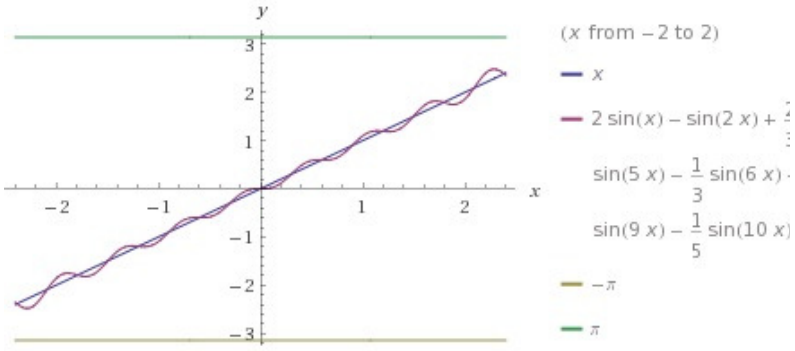
$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ... + (-1)^{n+1}\frac{\sin nx}{x}\right)$$

See graphs below of x and the approximation

$2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots + (-1)^{n+1}\frac{\sin nx}{x}\right)$ for $n = 1,\ 3,\ 10$







Example: in $C\{-\pi, \pi]$ let

$$W = \text{Span}\,\{1, x, x^2, x^3, x^4, x^5\}$$

What is proj_W sin ? (the best approximation of the sin function by a polynomial of degree ≤ 5?

Need to get an orthogonal basis: fair amount of work using Gram Schmidt.

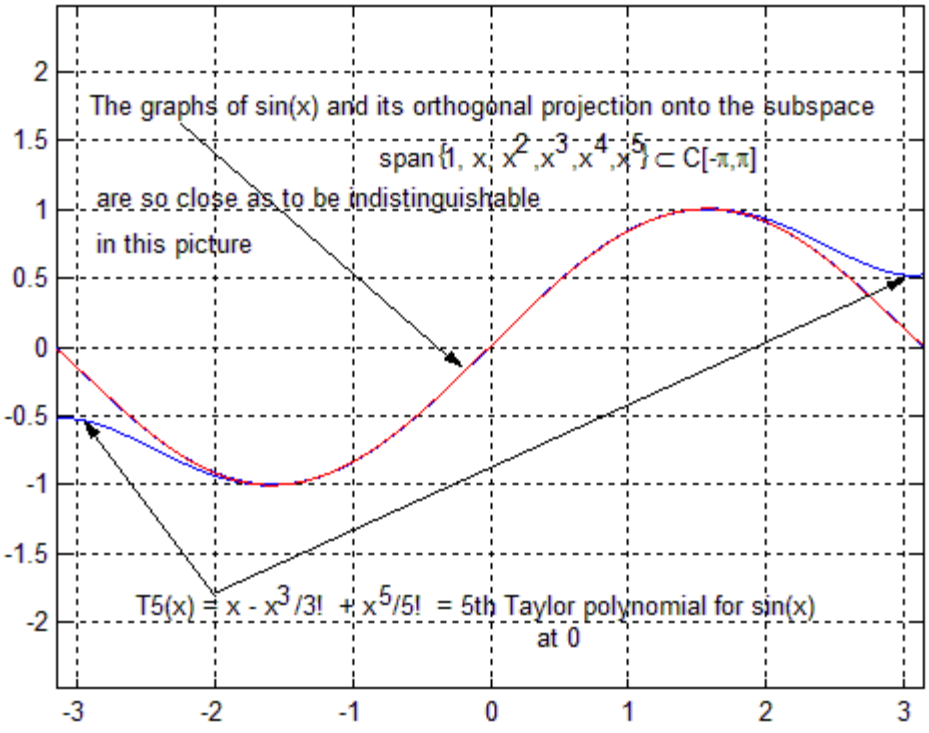
An orthonormal basis looks like

$$\{\ \tfrac{1}{\sqrt{2\pi}},\quad \tfrac{\sqrt{6}\,x}{2\pi\sqrt{\pi}},\quad \tfrac{1}{8}\tfrac{\sqrt{8}\,\sqrt{45}\,(x^2-\frac{1}{3}\pi^2)}{\sqrt{\pi^5}},\quad \ldots\ \}$$

Then (much more computation using this basis and the projection formula) proj_W sin $=$

$$\tfrac{21}{8\pi^{10}}\Big((5\pi^8 - 765\pi^6 + 7425\pi^4)x + (3750\pi^4 - 30\pi^6 - 34650\pi^2)x^3 + (33\pi^4 - 3465\pi^2 + 31185)x^5\Big)$$

Compare the graph with that of the 5^{th} Taylor polynomial for the sin function at 0.


The graphs of sin(x) and its orthogonal projection onto the subspace
span{1, x, x^2, x^3, x^4, x^5} ⊂ C[-π,π]
are so close as to be indistinguishable
in this picture
T5(x) = x - x^3/3! + x^5/5! = 5th Taylor polynomial for sin(x) at 0
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2
-1
0
1
2
3