Review: Basic Properties of inner products

$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

$\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} = (\boldsymbol{v} + \boldsymbol{w})\boldsymbol{u}$

$\boldsymbol{u} \cdot \mathbf{0} = 0$

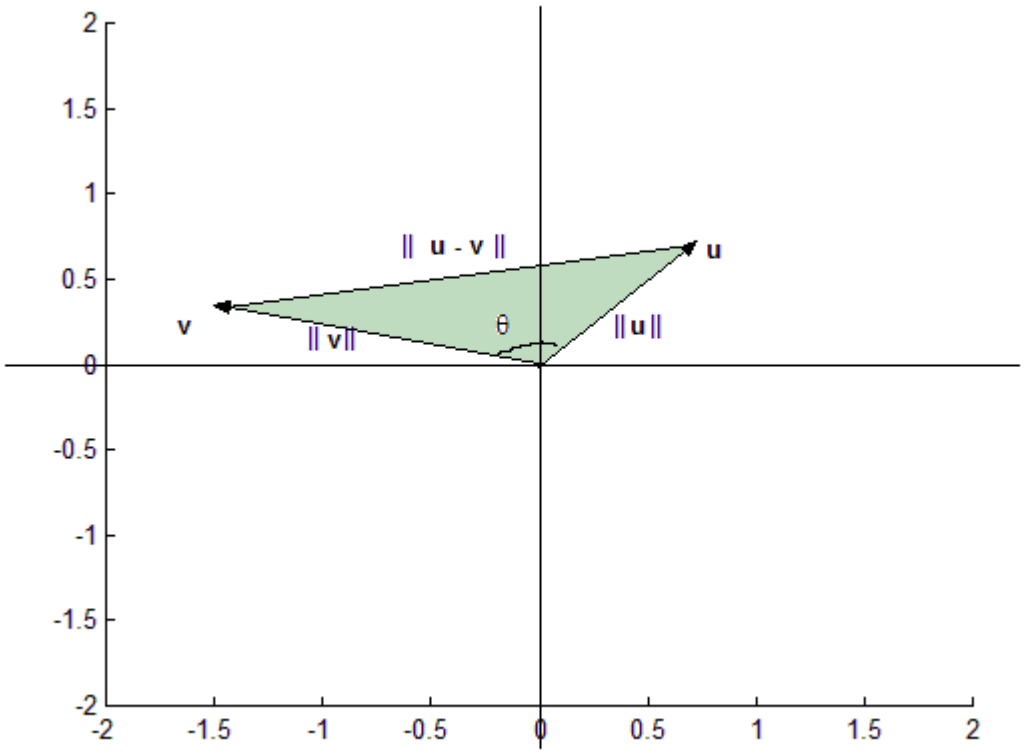
$(c\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$

$\boldsymbol{u} \cdot \boldsymbol{u} = u_1^2 + ... + u_n^2 \geq 0$
$\quad ||$
$||\boldsymbol{u}||^2$, so $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$

$\| \boldsymbol{u} - \boldsymbol{v} \|$ = Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$

$= \sqrt{(u - v) \cdot (u - v)}$

Angle θ between two vectors (by convention, choose $0 \le \theta \le \pi$)



$$||\boldsymbol{u}-\boldsymbol{v}||^2 = \begin{cases} \boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v} \;-\; 2\boldsymbol{u}\cdot\boldsymbol{v} \\ ||\boldsymbol{u}||^2+||\boldsymbol{v}||^2-2||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta \quad \text{(by Law of Cosines)} \end{cases}$$

$$= \begin{cases} \boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v} \;-2\; \boldsymbol{u}\cdot\boldsymbol{v} \\ \boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v} \;-2\; ||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\boldsymbol{\theta} \end{cases}$$

so $\quad \boldsymbol{u}\cdot\boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$

Example: Angle between $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$:

$$32 = \sqrt{14}\sqrt{77}\cos\theta$$

$$\cos\theta \approx 0.9746$$

$$\theta \approx 0.2257 \text{ (radians)} \approx 12.9332^\circ$$

Note, above, that $\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0$

$$\Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$$

Definition: Suppose $\boldsymbol{u}$, $\boldsymbol{v}$ are in $\mathbb{R}^n$. **We say** that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, written $\boldsymbol{u} \perp \boldsymbol{v}$ if $\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$

(So, by definition, $\mathbf{0} \perp \boldsymbol{v}$ for every vector $\boldsymbol{v}$)

Definition Let W be a subspoace of $\mathbb{R}^n$. Then $W^\perp = \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{z}$ is orthogonal to every vector in $W\}$.

$W^\perp$ is a subspace of $\mathbb{R}^n$; $W^\perp$ is called the orthogonal complement of W.

Example (see lecture natues and text) For an $m \times n$ matrix A :

$(\text{Row } A)^\perp = \text{Nul } A$ or, equivalently, $\text{Row } A = (\text{Nul } A)^\perp$

This is true for any matrix A, so, for example, we can replace A with A^T to get

$(\text{Row } A^T)^\perp = \text{Nul } A^T$ or, equivalently, $\text{Row } A^T = (\text{Nul } A^T)^\perp$

and, since $\text{Row } A^T = \text{Col } A$, we could alsow rrite

$(\text{Col } A)^\perp = \text{Nul } A^T$ or, equivalently, $\text{Col } A = (\text{nul } A^T)^\perp$

Definition $\mathcal{B} = \{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ...\boldsymbol{u_p}\}$ is called an orthogonal set if $\boldsymbol{u_i} \perp \boldsymbol{u_j}$ for all $i \neq j$

$\mathcal{B}$ is an orthonormal set if $\mathcal{B}$ is an orthogonal set and $||\boldsymbol{u_i}|| = 1$ for each i

An orthogonal set of nonzero vectors must be linearly independent. (*See discussion in textbook.*)

If $\mathcal{B} = \{\boldsymbol{u_1}, ..., \boldsymbol{u_i}, ..., \boldsymbol{u_p}\}$ and all $\boldsymbol{u_i} \neq \boldsymbol{0}$, then $\mathcal{B}$ is an orthogonal basis for the $W = \text{Span}\{\boldsymbol{u_1}, \boldsymbol{...}, \boldsymbol{u_p}\}$, a subspace W of $\mathbb{R}^n$

For $\boldsymbol{y} \in W$: $\boldsymbol{y} = \quad c_1\boldsymbol{u_1} \ + ... \ + ... \ c_i\boldsymbol{u_i} ... + ... + c_p\boldsymbol{u_p}$

Could set up and solve a sylatem of lienar equations to find the "unknown" coefficients c_i but finding the c_i's and be done more easily because $\mathcal{B}$ is an orthogonal basis (*see why in textbook*):

$$\begin{aligned} \boldsymbol{y} \quad &= \quad c_1 \ \boldsymbol{u_1} \ + ... \ + \ c_i \ \boldsymbol{u_i} \quad + ... \ + c_p \ \boldsymbol{u_p} \\ &= \quad \frac{\boldsymbol{y}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + ... \ + \frac{\boldsymbol{y}\cdot\boldsymbol{u_i}}{\boldsymbol{u_i}\cdot\boldsymbol{u_i}}\boldsymbol{u_i} \ + ... \ + \frac{\boldsymbol{y}\cdot\boldsymbol{u_p}}{\boldsymbol{u_p}\cdot\boldsymbol{u_p}}\boldsymbol{u_p} \end{aligned}$$

Even nicer (in some ways): if $\mathcal{B}$ is an orthonormal basis for subspace W, then each $\boldsymbol{u_i} \cdot \boldsymbol{u_i} = 1$, so the coefficient formulas simplify to $c_i = \frac{\boldsymbol{y}\cdot\boldsymbol{u_i}}{\boldsymbol{u_i}\cdot\boldsymbol{u_i}} = \boldsymbol{y} \cdot \boldsymbol{u_i}$
so

$$\boldsymbol{y} \quad = (\boldsymbol{y}\cdot\boldsymbol{u_1})\,\boldsymbol{u_1} \ + ... + \ (\boldsymbol{y}\cdot\boldsymbol{u_i})\,\boldsymbol{u_i} + ... + (\boldsymbol{y}\cdot\boldsymbol{u_p})\,\boldsymbol{u_p}$$

Example

$$\mathcal{B} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \{\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\boldsymbol{u_1}}, \underbrace{\begin{bmatrix}-1\\4\\1\end{bmatrix}}_{\boldsymbol{u_2}}, \underbrace{\begin{bmatrix}2\\1\\-2\end{bmatrix}}_{\boldsymbol{u_3}}\}$$

$=$ orthogonal basis for $W = \mathbb{R}^3$. To write

$$\boldsymbol{y} = \begin{bmatrix}1\\-2\\7\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\4\\1\end{bmatrix} + c_3\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Find

$$c_1 = \frac{\begin{bmatrix}1\\-2\\7\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}, \quad c_2 = \frac{\begin{bmatrix}1\\-2\\7\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\1\end{bmatrix}}{\begin{bmatrix}-1\\4\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\1\end{bmatrix}}, \quad c_3 = \frac{\begin{bmatrix}1\\-2\\7\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix}}{\begin{bmatrix}2\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix}}$$

$$c_1 = \tfrac{8}{2} = 4, \quad c_2 = \tfrac{-2}{18} = -\tfrac{1}{9}, \qquad c_3 = \tfrac{-14}{9}$$

so

$$\begin{bmatrix}1\\-2\\7\end{bmatrix} = 4\begin{bmatrix}1\\0\\1\end{bmatrix} - \tfrac{1}{9}\begin{bmatrix}-1\\4\\1\end{bmatrix} - \tfrac{14}{9}\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Let $\mathcal{B}' = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\} = \{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{9}} \\ \frac{1}{\sqrt{9}} \\ \frac{-2}{\sqrt{9}} \end{bmatrix},$

the same as $\mathcal{B} =$ except each that vector is normalized to have length 1: $\boldsymbol{w_i} = \frac{1}{||\boldsymbol{u_i}||}\boldsymbol{u_i}$

$\mathcal{B}'$ is an orthonormal basis for $\mathbb{R}^3$

Then $\boldsymbol{y} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3}$, where

$$c_1 = \boldsymbol{y} \cdot \boldsymbol{u_1} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{8}{\sqrt{2}}$$

$$c_2 = \boldsymbol{y} \cdot \boldsymbol{u_2} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} = \frac{-2}{\sqrt{18}} \quad \text{and}$$

$$c_3 = \boldsymbol{y} \cdot \boldsymbol{u_3} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{9}} \\ \frac{1}{\sqrt{9}} \\ \frac{-2}{\sqrt{9}} \end{bmatrix} = \frac{-14}{\sqrt{9}}$$

Notice that rescaling the vectors to get unit vectors in $\mathcal{B}'$ also rescaled the coefficients:

for basis $\mathcal{B}$: $\begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} + \ldots.$, but for basis $\mathcal{B}'$: $\begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \frac{8}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \ldots$

Factoring $\frac{1}{\sqrt{2}}$ out of the second vector turns the coefficient back into a 4

A moral: Normalizing the orthogonal base $\mathcal{B}$ to create an orthonormal base $\mathcal{B}$ ' makes the formula for the coefficients simpler; but the numerical vectors substituted into the formulas are more complicated (lots of $\sqrt{\ }$'s in the denominators). The orthonormal basis gives simpler coefficient formulas and is therefore nice for the theory and proving things, but it may make the numeric calculations uglier.