When is A diagonalizable?

($n \times n$ matrix)

We already know

A is diagonalizable

if and only if

There is a basis $\{v_1, \ldots, v_i, \ldots v_n\}$ for $\mathbb{R}^n$ where each basis vector v_i is an eigenvector of A

(So, it really depends on

"can we find enough linearly independent eigenvectors?"

Suppose A is 8×8 and that the characteristic polynomial is

$$C(\lambda) = (\lambda+1)(\lambda-1)^2(\lambda-3)^3(\lambda^2+\lambda+1) = 0$$

$(\lambda^2+\lambda+1)$: not factorable further using real numbers

$$= (\lambda - r_1)(\lambda - r_2) \text{ where } \begin{cases} r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \\ r_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\, i \end{cases}$$

(two more complex eigenvalues

Eigenvalues:	$\lambda_1 = -1$	$\lambda_2 = 1$	$\lambda_3 = 3$	$r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i$ $r_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$
Multiplicity of eigenvalue:	1	2	3	1 (each)
Eigenspace	E_1	E_2	E_3	E_4, E_5 ??
Dimension of eigenspace	?	?	?	?, ?

The Fundamental Theorem of Algebra says that $C(\lambda)$ will completely factor into linear factor if we allow complex numbers. For real numbers only, the Fundamental Theorem guarantees that the characteristic polynomial $C(\lambda)$ will factor completely into linear and irreducible quadratic factors such as $(\lambda^2+\lambda+1)$

Generally, in thsi course, were on interested in real $n \times n$ matrices A multiplying real vectors from $\mathbb{R}^n$. We could allow vectors with complex entries (vectors from $\mathbb{C}^n$) and matrices A with complex entries; we could then ask for a factorization $A = PDP^{-1}$. We would then also look at the complex eigenvalues. The same theory for finding eigenvalues, eigenvectors, doing the di\agonal factorization, etc. would work.

Usually, in this course, we are interested in the real eigenvalues.

In general, the theory goes: set the characteristic polynomial $= 0$

$$C(\lambda) = \det(A - \lambda I) = 0$$

and solve to find the eigenvalues of A

$(\lambda - \lambda_1)^{n_1} \ldots (\lambda - \lambda_i)^{n_i} \ldots (\lambda - \lambda_p)^{n_p}$ [****]

↑

[unfactorable part if using only real numbers]

Eigenvalues: $\lambda_1, \ldots, \lambda_i, \ldots, \lambda_p$
Multiplicity: $n_1, \ldots, n_i, \ldots, n_p$
Find each eigenspace by solving $E_1, \ldots, E_i, \ldots, E_p$

$$(A - \lambda_i I)\boldsymbol{x} = \mathbf{0}$$

Solution you a basis for eigenspace E_i, so you know dim E_i

New Information:

There is a basis $\{v_1, \ldots, v_i, \ldots v_n\}$ for $\mathbb{R}^n$ where each basis vector v_i is an eigenvector of A

if and only if

(sum of the dimensions of all eigenspaces) $= n$

To indicate why this is true, we gave a proof in class for when there are only two eigenspaces. Notes posted online soon will show the more general argument for those interested.

One other fact, not proven in class or text: notes online soon will give some indication about why it's true:

Theorem For each individal eigenvalue λ_i :

$$\dim(\text{eigenspace for } \lambda_i) \leq \text{ multiplicity of eigenvalue } \lambda_i$$

In the example above for an 8×8 matrix A for which:

Eigenvalues:	$\lambda_1 = -1$	$\lambda_2 = 1$	$\lambda_3 = 3$	$r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i$ $r_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$
Multiplicity of eigenvalue:	1	2	3	1 (each)
Eigenspace	E_1	E_2	E_3	E_4, E_5 ??
Dimension of eigenspace	?	?	?	?, ?

Assuming only real numbers are used:

$\dim(E_1) = 1$ (why not 0?)
$\dim(E_2) \leq 2$
$\dim(E_3) \leq 3$

The dimensions at up to at most 6, so the sum of dimensions can't be 8: A cannot be diagonalizable.

(*If we allowed the complex eigenvalues and eigenspaces, we would have*

$$dim(E_1) + dim(E_2) + dim(E_3) + dim(E_4) + dim(E_5) \leq 8$$

and the sum $= 8$ *if and only iuf each of the real eigenspaces has maximum possible dimension. So A might be diagonalizable: can't tell without more information to determine the exact dimensions of* $E_1,\ E_2,\ E_3$.

See text, p. 284, Example 4 for a 3×3 matrix where there are two eigenvalues, one with multiplicity 1, the other with multiplicity 2 but where both eigenspaces have dimension $1,$ so that A cannot be diagonalizable.

Summary Theorem About Diagonalization

Suppose A is an $n \times n$ matrix with eigenvalues
$\lambda_1, ..., \lambda_i, \ ..., \ \lambda_p$

1) A is diagonalizable if and only if there is a basis of eigenvectors $\{\boldsymbol{v_1}, ..., \boldsymbol{v_i}, ..., \boldsymbol{v_n}\}$ for $\mathbb{R}^n$.
Then $A = PDP^{-1}$, where
$P = [\boldsymbol{v_1} \ ... \ \boldsymbol{v_i} \ ... \ \boldsymbol{v_n}]$

D is diagonal matrix with the eigenvalues
$\lambda_1, ..., \lambda_i, ..., \lambda_n$ on its diagonal

2) If we choose a basis for each eigenspace $\mathcal{B}_1$,..., $\mathcal{B}_p$ and unite these bases into one collection $\mathcal{B}$, then $\mathcal{B}$ is a linearly independent set of eigenvectors in $\mathbb{R}^n$ – so $\mathcal{B}$ is a basis for $\mathbb{R}^n$ if the sum of the dimensions of the eigenspaces is n.

3) For each individual eigenvalue λ_i :

$$\dim(\text{eigenspace}) \leq \text{ multiplicity of eigenvalue}$$