$n \times n$ matrix A:

eigenvector $\boldsymbol{x}$: nonzero vector $\boldsymbol{x}$ so that $A\boldsymbol{x} = \lambda\boldsymbol{x}$
or $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$

λ is the eigenvalue for $\boldsymbol{x}$
λ might be 0: that's allowed

eigenvalues the solutions of the equation: characteristic polynomial of $A = 0$

$$\downarrow$$

$$\det(A - \lambda I) = 0$$

eigenvectors For each particular eigenvalue λ, find

$\text{Nul}(A - \lambda I)$ = set of solutions to

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}$$

$$\uparrow$$

This subspace of $\mathbb{R}^n$ =
the eigenspace for the eigenvalue λ

Eigenspace = {eigenvectors with eigenvalue λ) $\cup$ $\{\mathbf{0}\}$
($\mathbf{0}$ is included because the eigenspace is a vector subspace of $\mathbb{R}^n$, but $\mathbf{0}$ is not itself an eigenvector)

A can be factored as

(*) $A = PDP^{-1}$ (where D is diagonal)

if and only if

$\mathbb{R}^n$ has a basis consisting of
eigenvectors of A. In that case, the columns of P are eigenvectors in the bases for $\mathbb{R}^n$, and their eigenvalues (in same order) are the diagonal entries in D

Last lecture: we proved that

Theorem If $\boldsymbol{v_1}, ..., \boldsymbol{v_k}$ are eigenvectors that all have different eigenvalues, then $\{\boldsymbol{v_1}, ..., \boldsymbol{v_k}\}$ is a linearly independent set.

(*So, if there are n different eigenvalues, we get a linear independent set of n eigenvectors: these are automatically a basis for $\mathbb{R}^n$ so, in this case, A must be diagonalizable!*)

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \qquad \rightleftarrows \qquad (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$$

Find eigenvalues: $\det(A - \lambda I) = \boldsymbol{0}$

$\uparrow$

Characteristic Polynomial $= 0$

$$\det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$$

$||$

$$-(2-\lambda)(6+\lambda) - 9$$

$$= \lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3) = 0 \text{ so}$$

$$\boldsymbol{\lambda = -7 \text{ or } 3.}$$

Find corresponding eigenvectors:

Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ when $\lambda = -7$
and when $\lambda = 3$:

$$(A + 7I)\boldsymbol{x} = \boldsymbol{0}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix} \text{ so } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

(Rescaled, $\boldsymbol{x} = s \begin{bmatrix} -1 \\ 3 \end{bmatrix}$)

Eigenspace for $\lambda = -7$

$$(A - 3I)\boldsymbol{x} = \boldsymbol{0}$$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Eigenspace for $\lambda = 3$

In this example:

2×2 matrix A

2 distinct eigenvalues:
each eigenspace 1-dimensional (str line through $\mathbf{0}$ in $\mathbb{R}^2$)

Basis for eigenspaces:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

$\mathbb{R}^2$ has a basis of eigenvectors,

$\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ one from each eigenspace. We can get this by uniting the two bases: $\mathcal{B}_1 \cup \mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right.$

A is diagonalizable:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}^{-1}$$