A: $m \times n$ matrix, and

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ is } T(\boldsymbol{x}) = A\boldsymbol{x}$$

$$\dim \text{Nul } A = \text{number of nonpivot cols}$$

$$\dim \text{Col } A = \text{number of pivot columns}$$

so

$$\begin{array}{ccc} \textbf{dim Col}(\boldsymbol{A}) & + \textbf{dim}(\textbf{Nul } \boldsymbol{A}) = \boldsymbol{n} \\ \| & \| \\ \textbf{dim range}(\boldsymbol{T}) & + \textbf{dim ker}(\boldsymbol{T}) = \boldsymbol{n} \\ \| \textbf{ (def)} & \| \textbf{ (def)} \\ \textbf{rank}(\boldsymbol{A}) & + \textbf{nullity}(\boldsymbol{A}) = \boldsymbol{n} \end{array}$$

Rank Theorem $\longrightarrow$

$$A = \begin{bmatrix} 2 & -2 & 0 & 1 & 1 & 1 \\ 2 & -2 & 0 & 1 & 1 & 1 \\ 4 & -4 & 12 & 10 & 10 & 9 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 2 & -2 & 6 & 5 & 3 & 3 \end{bmatrix} \sim \dots$$

3 nonpivot columns (3 free variables)

$$\begin{bmatrix} 2 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{matrix} x_2 & & x_4 & & x_6 \\ \begin{bmatrix} \mathbf{1} & -1 & \mathbf{0} & \frac{1}{2} & \mathbf{0} & \frac{1}{8} \\ \mathbf{0} & 0 & \mathbf{1} & \frac{2}{3} & \mathbf{0} & \frac{1}{12} \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1} & \frac{3}{4} \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \end{matrix}$$

B (*an echelon form of* A) $\qquad$ U (*rref* A)

pivot columns of U ↑ ↑ ↑

pivot columns of A ↓ ↓ ↓

$$\begin{bmatrix} \mathbf{2} & -2 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ \mathbf{2} & -2 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ \mathbf{4} & -4 & \mathbf{12} & 10 & \mathbf{10} & 9 \\ \mathbf{0} & 0 & \mathbf{3} & 2 & \mathbf{1} & 1 \\ \mathbf{2} & -2 & \mathbf{6} & 5 & \mathbf{3} & 3 \end{bmatrix}$$

Basis for Col A
(dim Col $A = 3$)

Solution for $A\boldsymbol{x} = \mathbf{0}$: 3 free variables

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \boldsymbol{x_2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{x_4} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{x_6} \begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{1}{12} \\ 0 \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \textbf{Basis for Nul } \boldsymbol{A} \\ \textbf{(dim Nul } \boldsymbol{A}\textbf{)} = \mathbf{3} \end{array} \right. \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{1}{12} \\ 0 \\ -\frac{3}{4} \\ 1 \end{bmatrix} \right\}$$

Rank A + **dim Nul A**

(# of pivot cols) **(# of nonpivot nonpivot cols)**

$= \quad \mathbf{3} \quad + \quad \mathbf{3}$

$= \quad \mathbf{6} \quad$ **(total # of cols)**

A: $m \times n$ matrix, and

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ is } T(\boldsymbol{x}) = A\boldsymbol{x}$$

$$\begin{aligned} \dim \text{Nul } A &= \text{number of nonpivot cols} \\ \dim \text{Row } A &= \text{number of nonzero rows in an echelon form of } A \\ &\quad \| \\ \dim \text{Col } A &= \text{number of pivot columns of } A \end{aligned}$$

so

$$\begin{array}{ccc} \mathbf{dim\ Row}(A) & + \mathbf{dim}(\mathbf{Nul}\ A) & = n \\ \| & \| & \\ \mathbf{dim\ Col}(A) & + \mathbf{dim}(\mathbf{Nul}\ A) & = n \\ \| & \| & \\ \mathbf{dim\ range}(T) & + \mathbf{dim\ ker}(T) & = n \\ \| \textbf{(def)} & \| \textbf{(def)} & \\ \textbf{Rank Theorem} \to \ \mathbf{rank}(A) & + \mathbf{nullity}(A) & = n \end{array}$$

$\left\{ \begin{array}{l} \textbf{Basis for Row } \boldsymbol{A} \\ \textbf{(dim Row } \boldsymbol{A} \textbf{ = 3)} \end{array} \right.$

$$\begin{matrix} & & A & & & \\ \begin{bmatrix} 2 & -2 & 0 & 1 & 1 & 1 \\ 2 & -2 & 0 & 1 & 1 & 1 \\ 4 & -4 & 12 & 10 & 10 & 9 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 2 & -2 & 6 & 5 & 3 & 3 \end{bmatrix} & \sim \end{matrix}$$

$$\sim \begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{4} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

B (*an echelon form of* A)

$$\left\{ \begin{array}{l} [\mathbf{2} \quad \mathbf{-2} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1}], \\ \\ [\mathbf{0} \quad \mathbf{0} \quad \mathbf{3} \quad \mathbf{2} \quad \mathbf{1} \quad \mathbf{1}], [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{4} \quad \mathbf{3}] \end{array} \right\}$$

$$\left\{\begin{array}{l}\textbf{Basis for Col } A^T \\ \textbf{(dim Col } A^T = 3\textbf{)}\end{array}\right. \quad \left\{\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ 3 \end{bmatrix}\right\}$$

Two vectors $\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ are

called perpendicular (or orthogonal) if

$$\boldsymbol{x}^T \boldsymbol{r} = 0$$

$$\| $$

$$[x_1, x_2, \ ., x_n] \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$$= \quad [x_1 r_1 + x_2 r_2 + ... + x_n r_n] = [0] \text{ (or just } 0)$$

$\boldsymbol{x}^T \boldsymbol{r}$ is viewed as a scalar (though technically, in our definition, a 1×1 matrix)

$\boldsymbol{x}^T \boldsymbol{r}$ is called the inner product or scalar product or dot product if the vectors $\boldsymbol{x}$, $\boldsymbol{r}$.

(*We'll say more about this definition in Chapter 6, but it's a characterization of perpendicularity you say in Calc III.*)

Nul A is perpendicular to Row A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \text{Row}_1\ A \\ \text{Row}_2\ A \\ \vdots \\ \text{Row}_n\ A \end{bmatrix}$$

If $\boldsymbol{x}$ is in Nul A, then

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

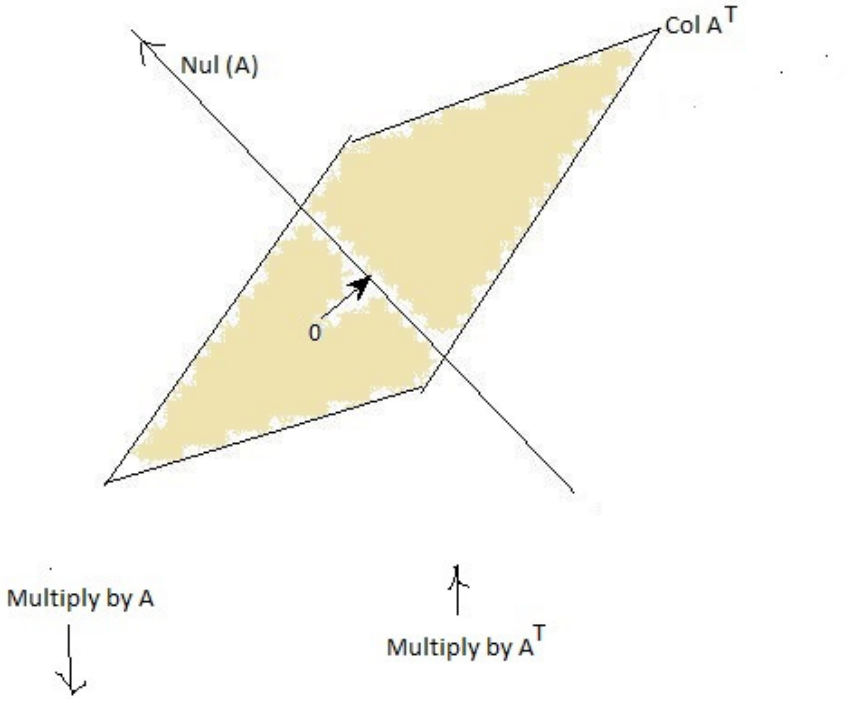
$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

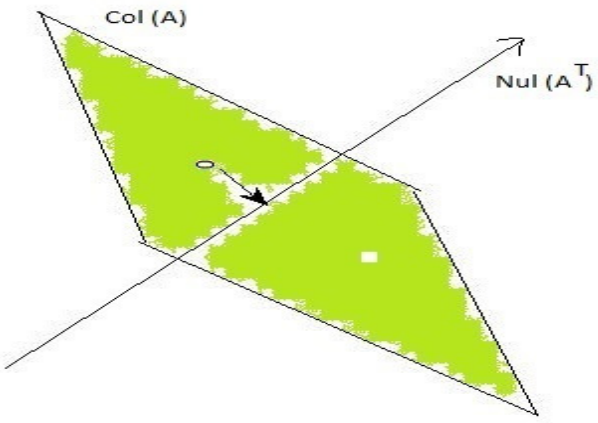
$$\begin{bmatrix} \boldsymbol{x}^T\text{Row}_1 A \\ \boldsymbol{x}^T\text{Row}_2 A \\ \vdots \\ \boldsymbol{x}^T\text{Row}_m A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ so $\boldsymbol{x}$ is perpendicular to

each vector in $\{\text{Row}_1 A, \dots, \text{Row}_m A\}$ and these vectors span $\text{Row}(A)$.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \;\; T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$





(OVER $\rightarrow$)

$\underline{\text{Nul}\,(A)}$ (Solve $A\boldsymbol{x} = \boldsymbol{0}$) $\quad \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 4 & 2 & 6 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\boldsymbol{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \qquad \text{(a straight line through } \boldsymbol{0}\text{)}$$

$$\underline{\text{Col}(A^T)} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$$

$||$ (a plane through $\boldsymbol{0}$)

also called $\text{Row}(A) = \underline{\text{the row space of } A}$

$\underline{\text{Informally note}}$: it appears that $\quad \begin{matrix} \text{Col}(A^T) \\ || \\ \text{Nul}(A) \perp \text{Row}\,(A) \end{matrix}$

$$\underline{\text{Col}(A)} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$$

$||$ (a plane through $\boldsymbol{0}$)

also called $\underline{\text{Row}(A^T)} = \underline{\text{the row space of } A^T}$

$\text{Nul}(A^T)$ (Solve $A^T\boldsymbol{x} = \boldsymbol{0}$) $\quad \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 4 & 2 & 6 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\boldsymbol{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{(a straight line through } \boldsymbol{0}\text{)}$$

$\text{Col}(A)$

Informally note: appears that $\text{Nul}(A^T) \perp \text{Row}\,(A^T)$ $\downarrow$