Practice

For a finite dimensional vector space V (one that has a finite spanning set), either

i) $V = \{\mathbf{0}\}$ and we say $\dim V = 0$

ii) If $V \neq \{\mathbf{0}\}$, the V has a basis and $\dim V =$ the number of elements in a basis (*we proved all bases for V contain <u>the same</u> number of vectors*)

Be cure you understand why:

$\dim \mathbb{R}^n =$ n

$\dim \mathbb{P}_n =$ $n+1$

$\dim M_{2,2} =$ 3

$\dim M_{m,n} =$ mn

$\dim \mathbb{P} =$ ∞

A is an $m \times n$ matrix, and

$T : \mathbb{R}^n \to \mathbb{R}^m$ is $T(\boldsymbol{x}) = A\boldsymbol{x}$

dim Col A = number of pivot columns
dim Nul A = number of nonpivot columns (= number of free variables in $A\boldsymbol{x} = \mathbf{0}$)

so

adding gets $\quad \mathbf{\dim \mathrm{Col}(A)} + \mathbf{\dim(\mathrm{Nul}\, A)} = \boldsymbol{n}$

$$\updownarrow \qquad\qquad \updownarrow$$

Rank Theorem: $\quad \mathbf{\mathrm{rank}(A)} + \mathbf{\mathrm{nullity}(A)} = \boldsymbol{n}$

$$\updownarrow \qquad\qquad \updownarrow$$

Rank Theorem (rephrased): $\quad \mathbf{\dim \mathrm{range}(T)} + \mathbf{\dim \ker(T)} = \boldsymbol{n}$

Example

A $\qquad\qquad$ B (*an echelon form of A*)

$$\begin{bmatrix} 2 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 4 & -4 & 12 & 10 & 10 & 9 \\ 2 & -2 & 0 & 1 & 1 & 1 \\ 2 & -2 & 6 & 5 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns $\quad \downarrow \qquad \downarrow \qquad \downarrow$

$$\sim \begin{bmatrix} \mathbf{1} & -1 & \mathbf{0} & \frac{1}{2} & \mathbf{0} & \frac{1}{8} \\ \mathbf{0} & 0 & \mathbf{1} & \frac{2}{3} & \mathbf{0} & \frac{1}{12} \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1} & \frac{3}{4} \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \quad (\textit{rref } A)$$

$\uparrow \qquad \uparrow \qquad \uparrow$

nonpivot columns (3 free variables)

$\boldsymbol{x_2} \qquad \boldsymbol{x_4} \qquad \boldsymbol{x_6}$

Basis for Col A ($\dim \mathrm{Col}\, A = 3$)

$$\begin{bmatrix} \mathbf{2} & -2 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ \mathbf{0} & 0 & \mathbf{3} & 2 & \mathbf{1} & 1 \\ \mathbf{4} & -4 & \mathbf{12} & 10 & \mathbf{10} & 9 \\ \mathbf{2} & -2 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ \mathbf{2} & -2 & \mathbf{6} & 5 & \mathbf{3} & 3 \end{bmatrix}$$

$\uparrow \qquad \uparrow \qquad \uparrow$

basis = the pivot columns of A

Solution for $A\boldsymbol{x} = \mathbf{0}$: 3 free variables corresppponding to nonpovito columns of A

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \boldsymbol{x_2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{x_4} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{x_6} \begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{1}{12} \\ 0 \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \textbf{Basis for Nul } \boldsymbol{A} \\ \textbf{(dim Nul } \boldsymbol{A}\textbf{)} = \mathbf{3} \end{array} \right. \qquad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{1}{12} \\ 0 \\ -\frac{3}{4} \\ 1 \end{bmatrix} \right\}$$

$$\begin{array}{ccccc} & \textbf{Rank } \boldsymbol{A} & + & \textbf{dim Nul } \boldsymbol{A} & \\ & \textbf{(\# of pivot cols)} & & \textbf{(\# of nonpivot nonpivot cols)} & \\ = & \mathbf{3} & + & \mathbf{3} & \\ = & \mathbf{6} & & & \textbf{(total \# of cols)} \end{array}$$

Practice:

Suppose $T : \mathbb{R}^5 \to \mathbb{R}^7$ is linear

the largest possible value for dim range$(T) =$ **5**

&

the corresponding value for dim ker$(T) =$ **0**
(*so T cannot be onto!*)

if the range of T is a plane through $\mathbf{0}$ in $\mathbb{R}^7$ **3**
what is dim$\{\boldsymbol{x} : T(\boldsymbol{x}) = \mathbf{0}\}$?

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear

If dim ker$(T) = 1$, then dim range $(T) =$ **2**

If dim ker$(T) = 2$, then dim range $(T) =$ **1**

If dim ker$(T) = 3$, then dim range $(T) =$ **0**, that is, range$(T) = \{\mathbf{0}\}$)

If dim **ker$(T) = 0$**, then dim range $(T) =$ (dimension of Range(T) would have to be 3, which can't be true since Range(T) is a subspace of $\mathbb{R}^2$)
↑
impossible !!
(*so T cannot be one-to-one !*)