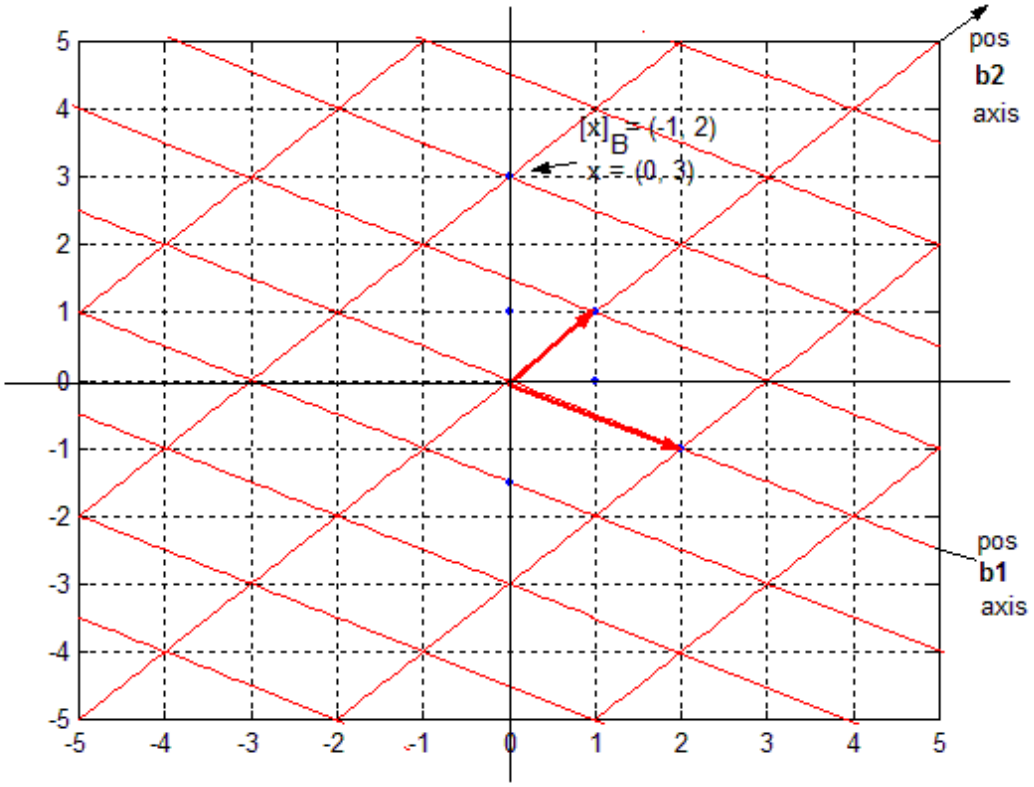
Use basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$



Geometrically: Where is $\boldsymbol{x}$ if $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$?

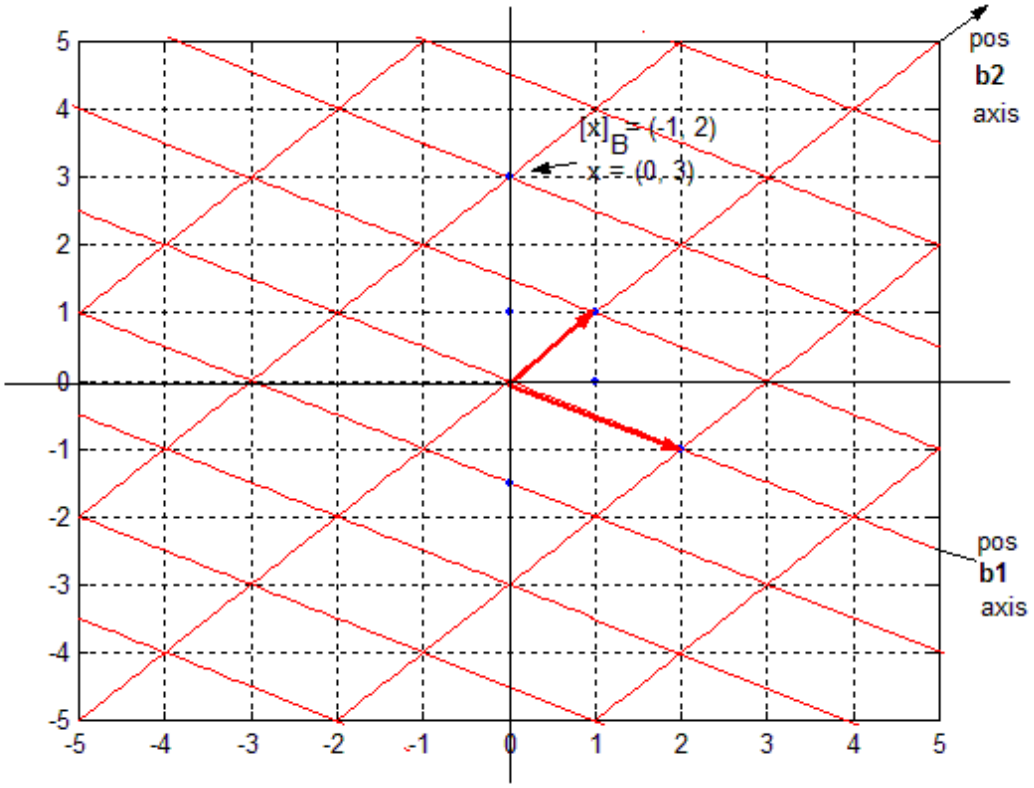
What are (approx.)the $\mathcal{B}$-coordinates for $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$?

$[\boldsymbol{b_1}]_{\mathcal{B}} = \begin{bmatrix} ** \\ ** \end{bmatrix}$ $[\boldsymbol{b_2}]_{\mathcal{B}} = \begin{bmatrix} ** \\ ** \end{bmatrix}$

Still using basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

$\boldsymbol{b_1}$ $\boldsymbol{b_2}$

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$ "change of coordinates matrix"



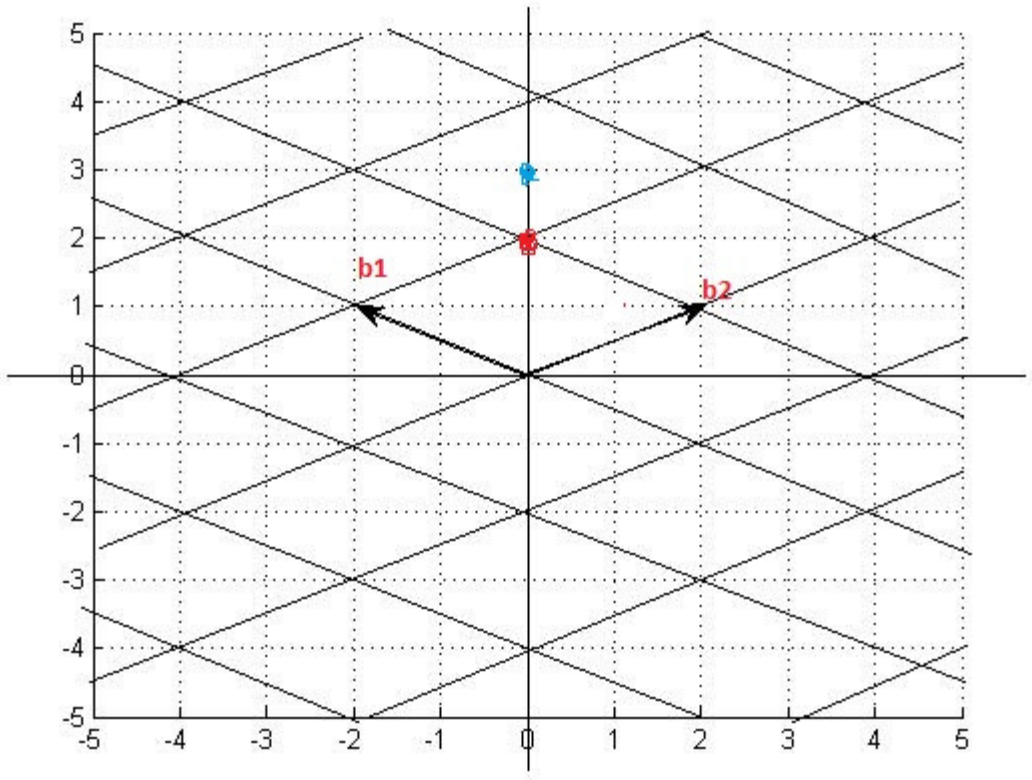
$$P_{\mathcal{B}} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ means ... ?

Multiplication by $P_{\mathcal{B}}$ converts <u>from $\mathcal{B}$-coordinates to</u> standard coordinate.

Why is $P_{\mathcal{B}}^{-1}$ invertible?

$$P_{\mathcal{B}}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ means ... ?

Multiplication by $P_{\mathcal{B}}^{-1}$ converts from <u>standard coordinates to $\mathcal{B}$-coordinates</u>



$A = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible, so its columns are a basis

$$\mathcal{C} = \left\{ \begin{matrix} \uparrow & \uparrow \\ \boldsymbol{c_1}, & \boldsymbol{c_2} \end{matrix} \right\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

The change of coordinates matrix $P_{\mathcal{C}} = A$

Any invertible matrix can be thought of as being the change of coordinates matrix (for the basis consisting of its columns)

Meaning (*verify in the picture)* :

$$P_{\mathcal{C}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad P_{\mathcal{C}} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \text{ (standard coordinates)}$$

$$P_{\mathcal{C}}^{-1} P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}} = P_{\mathcal{C}}^{-1}\boldsymbol{x} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = [\boldsymbol{x}]_{\mathcal{C}}$$

Multiplication by $P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$ converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates

Rotating the Coordinate System

ϕ = some angle (fixed)

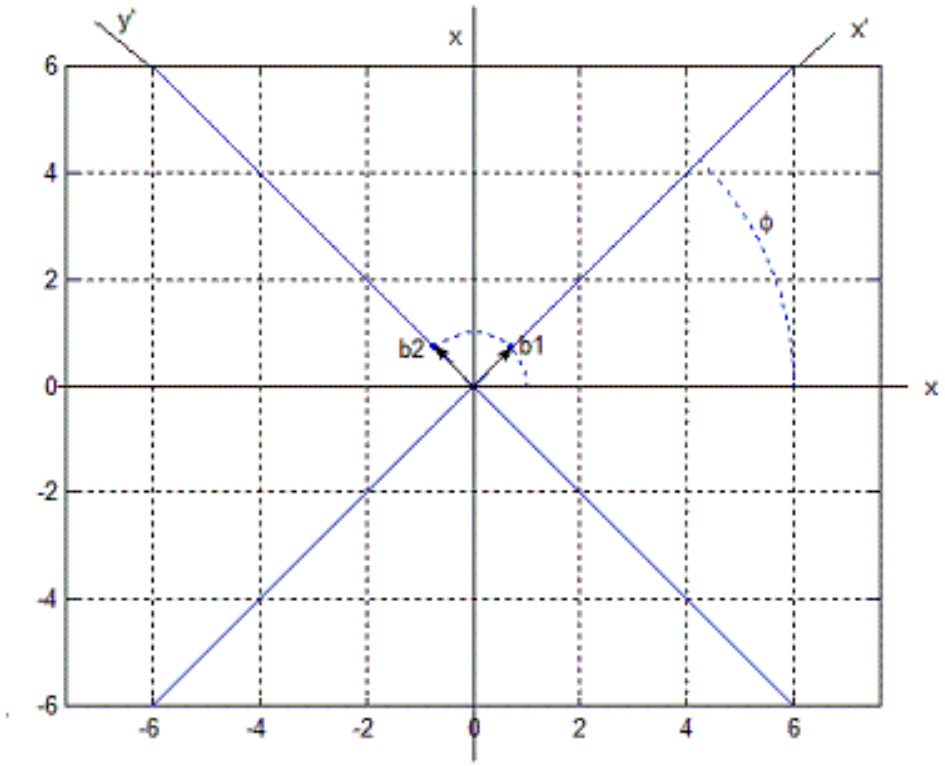
Rotation matrix $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is invertible (because determinant $\neq 0$)

so its columns $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ are linearly independent, span $\mathbb{R}^2$ and can be used as a basis.

$$\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \{\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}\} \text{ is a basis for } \mathbb{R}^2$$

Call the new coordinate axes (determined by $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ the x' and y' axes: so $\mathcal{B}$-coordinates are the same thing as x'-y' coordinates:

$$[\boldsymbol{x}]_{\boldsymbol{\mathcal{B}}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Converting $\begin{bmatrix} x' \\ y' \end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix}$

$\mathcal{B}$-coordinates $\leftrightarrow$ standard coordinates

$$\underset{\uparrow \boldsymbol{b_1}}{\cos\phi} \quad \underset{\uparrow \boldsymbol{b_2}}{-\sin\phi} \qquad \underset{P_{\mathcal{B}}}{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \underset{[\boldsymbol{x}]_{\mathcal{B}}}{\begin{bmatrix} x' \\ y' \end{bmatrix}} = \underset{\boldsymbol{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} \qquad (*)$$

so $$\begin{cases} x = x'\cos\phi - y'\sin\phi \\ y = x'\sin\phi + y'\cos\phi \end{cases}$$

In the other directionm

$$\underset{P_{\mathcal{B}}^{-1}}{\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{[\boldsymbol{x}]_{\mathcal{B}}}{\begin{bmatrix} x' \\ y' \end{bmatrix}} \qquad (**)$$

so $$\begin{cases} x' = \quad x\cos\phi + y\sin\phi \\ y' = -x\sin\phi + y\cos\phi \end{cases}$$

What does the graph of $5x^2 - 4xy + 5y^2 = 63$
look like ?

Rotate axes counterclockwise by $\frac{\pi}{4}$ and convert to x'-y' coordinates
(*don't worry about how $\frac{\pi}{4}$ was chosen*)

$$\begin{cases} x = x'\cos\phi - y'\sin\phi \\ y = x'\sin\phi + y'\cos\phi \end{cases}$$

$$\begin{aligned} 63 \quad &= 5x^2 - 4xy + 5y^2 \\ &= \quad 5(x'\cos\phi - y'\sin\phi)^2 \\ &\quad - 4(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) \\ &\quad + 5(x'\sin\phi + y'\cos\phi)^2 \end{aligned}$$

With $\phi = \frac{\pi}{4}$, we get

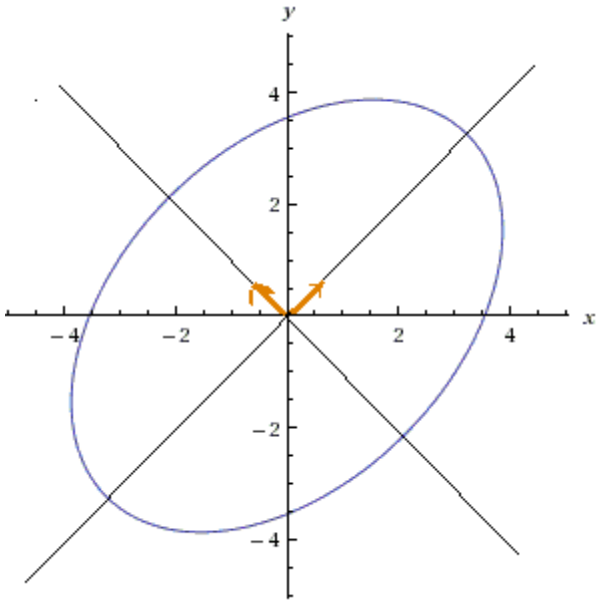
$$\begin{aligned} 63 \quad = \quad & 5(x'\tfrac{\sqrt{2}}{2} - y'\tfrac{\sqrt{2}}{2})^2 \\ & - 4(x'\tfrac{\sqrt{2}}{2} - y'\tfrac{\sqrt{2}}{2})(x'\tfrac{\sqrt{2}}{2} + y'\tfrac{\sqrt{2}}{2}) \\ & + 5(x'\tfrac{\sqrt{2}}{2} + y'\tfrac{\sqrt{2}}{2})^2 \end{aligned}$$

Simplifying gives

$$3(x')^2 + 7(y')^2 = 63, \text{ or}$$

$$\text{or } \frac{(x')^2}{21} + \frac{(y')^2}{9} = 1$$

See grsaph on folllowing page. Rotating the coordinate system (chaning to x'-y' coordinates has put the equation in a new recognizable form: the graph is an ellipse.



$$\frac{(x')^2}{21} + \frac{(y')^2}{9} = 1$$

$$5x^2 - 4xy + 5y^2 = 63$$

What does $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ mean

geometrically?

--A rotation (by $\frac{\pi}{2}$) that “moves” points

--A change of coordinates (no points “move”)