The Spanning Set Theorem (*review from last lecture*)

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$

$H = \text{Span } S = \text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$

Assume $H \neq \{\mathbf{0}\}$ (*which could only happen if all the vectors* $\boldsymbol{v_1} = \boldsymbol{v_2} = \ldots = \boldsymbol{v_p} = \mathbf{0}$)

Then 1) If one vectors in S is a linear combination of the others, we can delete it and get a subset (one vector smaller) $S' \subseteq S$ that has the same span :

$$\text{Span } S' = \text{Span } S = H$$

2) Some subset of S is a basis for H.

True/False: Practice

1. If A is an invertible $n \times n$ matrix, then the columns of A for a basis for $\mathbb{R}^n$

2. The vector space $V = \{\mathbf{0}\}$ has a basis $\mathcal{B} = \{\mathbf{0}\}$.

3. Suppose vector space $V \neq \{\mathbf{0}\}$. A basis for V is a linearly independent set that is as large as possible.

4. Suppose vector space $V \neq \{\mathbf{0}\}$. A basis for V is a spanning set that is as small as possible.

5. Suppose $\mathcal{B} = \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_p}\}$ is a basis for V and that $\boldsymbol{x} \in V$.

If $\begin{cases} \boldsymbol{x} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} + \ldots + c_p\boldsymbol{b_p} \\ \quad \text{and} \\ \boldsymbol{x} = d_1\boldsymbol{b_1} + d_2\boldsymbol{b_2} + \ldots + d_p\boldsymbol{b_p} \end{cases}$

then $c_1 = d_1,\ c_2 = d_2,\ \ldots\ ,\ c_p = d_p$

Basis for Col A = {*pivot columns of* A}
(where A is not the zero matrix) *Why?*

$$\underset{A}{[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_i}\ \ldots\ \boldsymbol{a_n}]} \sim \ \ldots\ \sim \underset{U=\text{rref}(A)}{[\boldsymbol{u_1}\ \boldsymbol{u_2}\ \ldots \boldsymbol{u_i}\ \ldots \boldsymbol{u_n}]}$$

A and U have the same pivot and nonpivot columns. Suppose $\boldsymbol{u_i}$ and $\boldsymbol{a_i}$ are corresponding nonpivot columns

1) First look at the simpler matrix U. Each nonpivot column of U is a linear combination of pivot columns in U: why?

Each nonzero entry in column $\boldsymbol{u_i}$ is not a pivot, so for that entry

i) in same row, there is a pivot **1** in a pivot column to the left) and, of course, all other entries in that pivot column are 0
ii) $\boldsymbol{u_i}$ is a weighted sum of these pivot columns. For example

$$U = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{3} & 0 & \mathbf{3} & 0 \\ 0 & \mathbf{1} & \mathbf{2} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2) It follows that the same is true in A: each nonpivot column of A is a linear combination of pivot columns. Why?

If $\boldsymbol{u_i}$ is a nonpivot column in U, then by 1), $\boldsymbol{u_i}$ can be written as a linear combo of pivot columns:

$c_1\boldsymbol{u_1} + \ldots + \cancel{c_i\boldsymbol{u_i}} + \quad \ldots \quad + c_n\boldsymbol{u_n} = \boldsymbol{u_i}$ where the weights are 0 for nonpivot columns

$$c_1\boldsymbol{u_1} + \ldots + (-1)\boldsymbol{u_i} + \ldots + c_n\boldsymbol{u_n} = \mathbf{0}, \text{ so } U\begin{bmatrix} c_1 \\ \vdots \\ -1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$$

.But $U\boldsymbol{x} = 0$ and $A\boldsymbol{x} = \mathbf{0}$ have the same solutions, so $A\begin{bmatrix} c_1 \\ \vdots \\ -1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$,

so $c_1\boldsymbol{a_1} + \ldots + (-1)\boldsymbol{a_i} + \ldots + c_n\boldsymbol{a_n} = \mathbf{0}$

so $c_1\boldsymbol{a_1} + \ldots \quad \cancel{-}\,\cancel{\boldsymbol{a_i}} \quad + \ldots + c_n\boldsymbol{a_n} = \boldsymbol{a_i}$

where the weights are 0 for nonpivot columns

so the nonpivot column $\boldsymbol{a_i}$ is a linear combo of pivot columns of A.

3) The pivot columns of A are linearly independent because: if we delete the nonpivot columns, the matrix B that remains has only pivot columns, so $B\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. Si the columns of B are linearly independent..

So if we start with $A = [\boldsymbol{a_1}\ \boldsymbol{a_2} \ldots\ \boldsymbol{a_n}]$

then $\text{Col}(A) = \text{Span}\{\boldsymbol{a_1}, \ldots, \boldsymbol{a_i}, \ldots, \boldsymbol{a_n}\}$

Since each nonpivot column is a linear combination of pivot columns, we can use the Spanning Set Theorem and delete the nonpivot columns, one at a time, leaving a set that still spans $\text{Col}(A)$.

$$\begin{aligned}
\text{Col}(A) &= \text{Span}\{\boldsymbol{a_1}, \ldots, \boldsymbol{a_i}, \ldots, \boldsymbol{a_n}\} \\
&= \text{Span}\{\boldsymbol{a_1}, \ldots, \cancel{\boldsymbol{a_i}}, \ldots, \boldsymbol{a_n}\} \\
&= \ \vdots\ \text{continue, delete all nonpivot columns deleted)} \\
&= \text{Span}\{\text{pivot columns of } A\} \\
&\qquad\qquad\qquad\qquad \uparrow \\
&\qquad\qquad\qquad \text{linearly independent}
\end{aligned}$$

so $\{$pivot columns of $A\}$ = a basis for $\text{Col}(A)$

Example

Pivot Columns (detemined by row reducing and looking at U)

Nonpivot columns

$$\begin{matrix} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{a_6} \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & 3 & 0 \\ 1 & 1 & 5 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 \\ -1 & 0 & -3 & -1 & -5 & 1 \\ 2 & 2 & 10 & 0 & 8 & 0 \\ 5 & 3 & 21 & 1 & 20 & 0 \end{bmatrix}$$

U = row reduced echelon form of A

$$\begin{matrix} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \boldsymbol{u_4} & \boldsymbol{u_5} & \boldsymbol{u_6} \end{matrix}$$

$$U = \begin{bmatrix} 1 & 0 & 3 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

The pivot columns of A are a basis for Col A

Note: Cannot use the pivot columns of U as a base for Col (A).
Look at preceding example: pivot columns of U are a basis for col(U)

but *Column 1 of* $A = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}$ is in Col(A) and cannot be written as a linear

combination of columns of U (all of which have bottom entry $= 0$).

After row reduction: Row$(A) \neq$ Col(U).

the pivot columns of U.

Example $A = \begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix}$

$$A \sim U = \begin{bmatrix} \mathbf{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(not linearly independent set: why?)

The pivot columns of A are columns $1, 3,$ and 5

A basis for col A is $\left\{ \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$

Side comment: In this example, we could easily give a simpler basis for col(A) : A has a pivot in every row, so Col $A = \mathbb{R}^3$. Knowing this we could say instantly that that $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ is a basis for col A.

Coordinates in a vector space V
(See lecture)

For $V = \mathbb{P}_2$, choose basis $\mathcal{B} = \{1, \quad t, \quad t^2\}$ (*the standard basis for* $\mathbb{P}_2$)

$\mathbb{P}$		$\mathbb{R}^3$
$\boldsymbol{p}$	$\mapsto$	$[\boldsymbol{p}]_{\mathcal{B}}$

This mapping is a 1-1 linear transformation from $\mathbb{P}$ to $\mathbb{R}^n$ (an isomorphism of vecgor soaces.)

$\boldsymbol{p}$ $\quad \boldsymbol{p}(t) = 2 + 3t - t^2 \mapsto \quad [\boldsymbol{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$\boldsymbol{q}$ $\quad \boldsymbol{q}(t) = -1 + 2t + t^2 \quad \mapsto \quad [\boldsymbol{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\mathbf{0}$ $\quad \mathbf{0}(t) = 0 + 0t + 0t^2 \quad \mapsto \quad [\mathbf{0}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boldsymbol{p} + \boldsymbol{q}$

$(\boldsymbol{p} + \boldsymbol{q})(t) = 1 + 5t \quad \mapsto \quad [\boldsymbol{p} + \boldsymbol{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$

$= [\boldsymbol{p}]_{\mathcal{B}} + [\boldsymbol{q}]_{\mathcal{B}}$

$c\boldsymbol{p}$

$(c\boldsymbol{p})(t) = 2c + 3ct - ct^2 \mapsto \quad [c\boldsymbol{p}]_{\mathcal{B}} = \begin{bmatrix} 2c \\ 3c \\ -c \end{bmatrix} = c\,[\boldsymbol{p}]_{\mathcal{B}}$

$3\boldsymbol{p} - 2\boldsymbol{q} = \mathbf{0}$ $\quad \Leftrightarrow \quad [3\boldsymbol{p} - 2\boldsymbol{q}]_{\mathcal{B}} = 3[\boldsymbol{p}]_{\mathcal{B}} - 2[\boldsymbol{q}]_{\mathcal{B}} = [\mathbf{0}]_{\mathcal{B}}$

(false)

$$3\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \textit{(false)}$$

Are $\quad \boldsymbol{p}(t) = 1 + 2t,$
$\boldsymbol{q}(t) = -2 + 3t^2,$ and
$\boldsymbol{r}(t) = 1 + 2t - t^2$ linearly independent vectors in $\mathbb{P}_2$???

If $c_1p(t) + c_2q(t) + c_3r(t) = 0$, **must** $c_1 = c_2 = c_3 = 0$**?**

Use the coordinate mapping $[\ ]_{\mathcal{B}}$ to "translate" to <u>equivalent</u> question about coordinate vectors in $\mathbb{R}^3$:

If $c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\0\\3\end{bmatrix} + c_3\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, **must** $c_1 = c_2 = c_3 = 0$**?**

Yes:

$$\begin{bmatrix}1 & -2 & 1\\2 & 0 & 2\\0 & 3 & -1\end{bmatrix} \sim \ldots \sim \begin{bmatrix}1 & -2 & 1\\0 & 4 & 0\\0 & 0 & -1\end{bmatrix}$$

(3 pivot positions)