Some definitions and theorems – all these were true in $\mathbb{R}^n$ and are equally true for any vector space V

Suppose $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p} \in V$ where V is a vector space

$\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}$ are **linearly independent** means:

$c_1\boldsymbol{v_1} + ... + c_p\boldsymbol{v_p} = \mathbf{0}$ has only the trivial solution $c_1 = ... = c_p = 0$

So **linearly dependent** means:

$c_1\boldsymbol{v_1} + ... + c_p\boldsymbol{v_p} = \mathbf{0}$ has a nontrivial solution (not all of $c_1, ..., c_p$ are 0)

Examples (reminders)

$\{\mathbf{0}\}$ is linearly dependent

$\{\boldsymbol{v_1}\}$ is linearly independent if $\boldsymbol{v_1} \neq \mathbf{0}$

$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is linearly independent if and only if neither vector if a scalar multiple of the other

For $p \geq 2$:

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ is linearly dependent if and only if at least one vector can be qritten as a linear combination of the others.

"Preceding Vectors" Theorem: Suppose $\boldsymbol{v_1} \neq \mathbf{0}$

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_i}, \ldots, \boldsymbol{v_p}\}$ is linearly dependent if and only if **one of the vectors** is a linear combination of the preceding vectors.

Example

$M_{22} =$ the vector space consisting of all 2×2 matrices. Let

$$\mathcal{B} = \left\{ \underset{\boldsymbol{v_1}}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{\boldsymbol{v_2}}{\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}}, \underset{\boldsymbol{v_3}}{\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}}, \underset{\boldsymbol{v_4}}{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}} \right\}$$

$\mathcal{B}$ is linearly independent: To see this, suppose

$$c_1 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so

$$\begin{bmatrix} 2c_1 + c_2 + 2c_4 & 6c_2 + c_3 \\ 3c_3 & c_3 + 4c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2c_1 + c_2 + 2c_4 = 0 \\ 6c_2 + c_3 = 0 \\ 3c_3 = 0 \\ c_3 + 4c_4 = 0 \end{cases} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0$$

Also, Span $\mathcal{B} =$ Span$\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \right\} = M_{22}$

GTo see this, consider any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in M_{22}. Can we solve

$$c_1 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It a solution because

$$\begin{bmatrix} 2 & 1 & 0 & 2 & a \\ 0 & 6 & 1 & 0 & b \\ 0 & 0 & 3 & 0 & c \\ 0 & 0 & 1 & 4 & d \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

(a consistent system)

For a specific example, given $\begin{bmatrix} \frac{5}{2} & -7 \\ 3 & -\frac{1}{9} \end{bmatrix}$: solving the system gives

$$\frac{79}{36}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} - \frac{5}{18}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -7 \\ 3 & -\frac{1}{9} \end{bmatrix}$$

In summary: $\mathcal{B} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \right\}$ is

a linearly independent set in M_{22} that also spans M_{22}

Definition Suppose H is a subspace of vector space V (possibly $H = V$)

$\mathcal{B} = \{ \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p} \}$ is called a **basis for H** if

1) $H = \text{Span}\{ \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p} \}$

Notice each $\boldsymbol{v_i}$ is automatically in H because
$\boldsymbol{v_i} = 0 \cdot \boldsymbol{v_1} + 0 \cdot \boldsymbol{v_2} + ... + 1 \cdot \boldsymbol{v_i} + ... + 0 \cdot \boldsymbol{v_n}$

2) $\{ \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p} \}$ **is linearly independent**

Intuitively, a basis $\{ \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p} \}$ for H is
"just the right size:"

"big enough" – every $\boldsymbol{v}$ in H can be "built" as a linear combo of $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}$

"but not too big" – no $\boldsymbol{v_i}$ could be deleted and;
for a basis, $H \neq \text{Span}\{ \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \cancel{\boldsymbol{v_i}}, ..., \boldsymbol{v_p} \}$

Example Above, $\mathcal{B}$ is a basis for M_{22}

Many other examples in the lecture

The Spanning Set Theorem

Let $S = \left\{ \boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \right\}$

$H = \text{Span } S = \text{Span} \left\{ \boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \right\}$

Assume $H \neq \{\mathbf{0}\}$ (*this could only happen if all vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ *were* $\mathbf{0}$)

Then

1) If one of the vectors is a linear combination of the others, we can delete it and get a (one vector smaller) subset $S' \subseteq S$ that has the same span :

$$\text{Span } S' = \text{Span } S = H$$

and 2) Some subset of S is a basis for H

(*discussed in class: if* S *is not already linearly independent, delete one at a time vectors that can be written as a linear combination of the others. By part 1), each deletion does not change the span. Eventually,* S *is reduced to a subset* S' *that still spans and is also linearly independent.* S' *is a basis for* H)