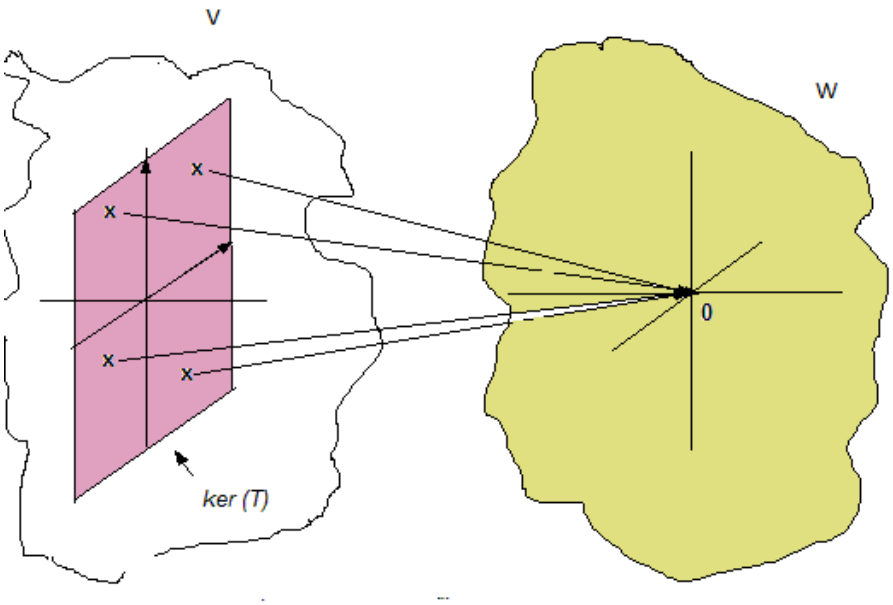
$T : V \to W$ linear transformation

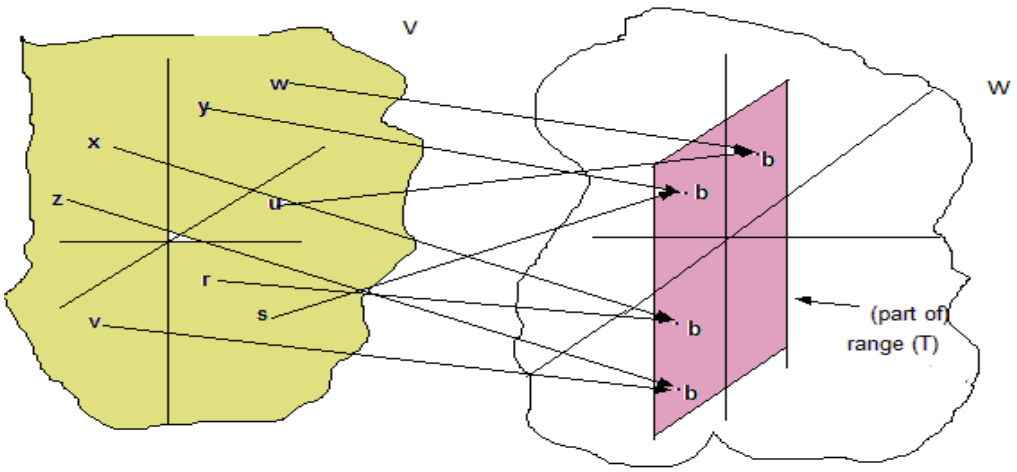


↑

Kernel of $T = \ker(T)$ is a subspace of V

$\text{range}(T)$ is a subspace of W

↓



Special Case where $T : \mathbb{R}^n \to \mathbb{R}^m$ (linear)

$$\begin{matrix} \mathbb{R}^n & & \mathbb{R}^m \\ || & & || \\ V & & W \end{matrix}$$

Then there is an $m \times n$ matrix A

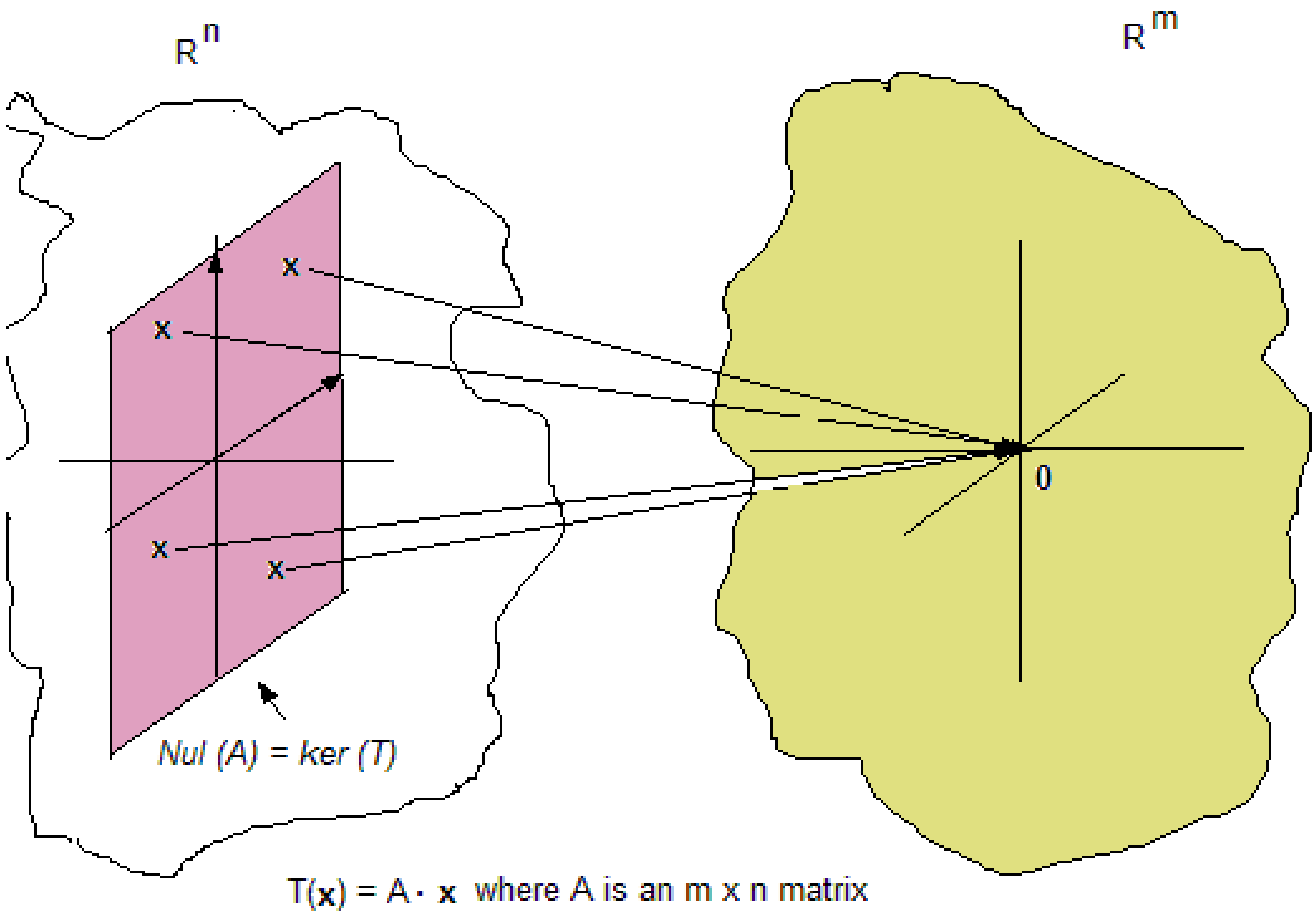
$$T(\boldsymbol{x}) = A\boldsymbol{x}$$

Kernel of T = $\ker T = \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \mathbf{0}\}$

also called

Null space of A = Nul A $= \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \mathbf{0}\}$

↓



Example $T : \mathbb{R}^5 \to \mathbb{R}^3$

$$T(\boldsymbol{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + 10x_2 + 2x_3 + 9x_4 + 4x_5 \\ 3x_1 + 6x_2 + x_3 + 5x_4 + x_5 \\ 2x_1 + 4x_2 + x_3 + 4x_4 + 2x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix} \boldsymbol{x}$$

Is some particular vector in Nul A ? EASY

Is $\begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ in ker(T) = Nul(A)?

Easy: just check whether

$$\underset{A}{\begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}} = \underset{\mathbf{0?}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \quad \text{yes}$$

Is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in ker(T) = Nul(A)?

$$\begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 25 \\ 15 \\ 10 \end{bmatrix}, \text{ no}$$

Harder: Describe Nul(A) explicitly
(give a "recipe" for the vectors in Nul(A))

Solve $Ax = 0$:

i) only solution $\boldsymbol{x} = \mathbf{0}$? Then Nul$(A) = \{\mathbf{0}\}$

ii) otherwise: write solutions in parametric vector form using free variables as parameters *(as usual)*

$$\underset{A}{\begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix}} \sim \ldots \sim \underset{\substack{x_2 \\ \downarrow}}{\begin{bmatrix} \mathbf{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}} \underset{\substack{x_4 \text{ free} \\ \downarrow}}{}, \text{ so}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Therefore $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

This always works to describe Nul A: Solve $A\boldsymbol{x} = \mathbf{0}$

i) If only solution is $\boldsymbol{x} = \mathbf{0}$, then $\text{Nul}(A) = \{\mathbf{0}\}$

ii) Otherwise, there are free variables and infinitely many solutions. Write the general solution in parametric vector form using the free variables as the parameters. Then $\text{Nul}(A)$ is the span of the column vectors involved: $\text{Nul}(A) = \text{Span}\left\{ *, *, \ldots, * \right\}$

where the number of vectors $*, *, \ldots, *$ = the number of free variables.

A nice observation on the side (for free): these vectors $*, *, \ldots, *$ are automatically linearly independent.

To illustrate, suppose there were (say) 3 free variables: say, x_1, x_4, x_5. Look at

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ ****** \\ ****** \\ x_5 \\ x_5 \\ ****** \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ * \\ * \\ 0 \\ 0 \\ * \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ * \\ * \\ 1 \\ 0 \\ * \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ * \\ * \\ 0 \\ 1 \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A linear combination of these vectors can be $\mathbf{0}$ only if all of x_1, x_4, x_5 are 0.

Range of T :

Special Case where $T : \mathbb{R}^n \to \mathbb{R}^m$ (linear)

$$\begin{matrix} \mathbb{R}^n & & \mathbb{R}^m \\ \| & & \| \\ V & & W \end{matrix}$$

Then there is an $m \times n$ matrix $A = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_n}]$

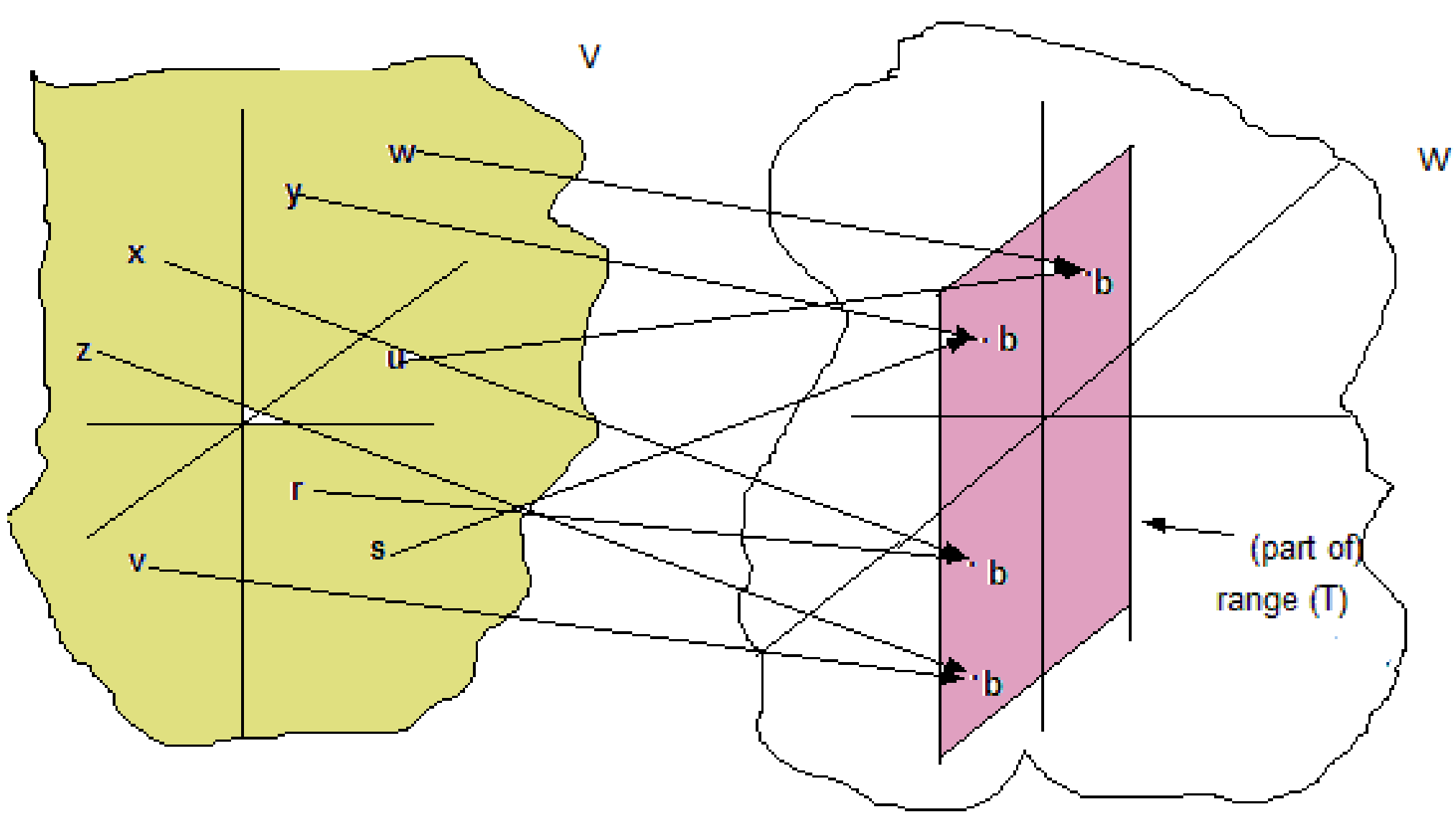
$$T(\boldsymbol{x}) = A\boldsymbol{x}$$

Range of T $= \{\boldsymbol{b} \in \mathbb{R}^m : T(\boldsymbol{x}) = \boldsymbol{b} \text{ has a solution}\}$
$= \{\boldsymbol{b} \in \mathbb{R}^m : T(\boldsymbol{x}) = \boldsymbol{b} \text{ has a solution}\}$
$= \text{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, .., \boldsymbol{a_n}\}$

also called

$=$ Column Space of A $= \text{Col}(A)$

↓



Explicitly describe the column space of A: easy

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\boldsymbol{x} \in \text{Col}(A)$ if and only if $\boldsymbol{x}$ is a linear combination of these columns:

$$\boldsymbol{x} = c_1 \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} + ... + c_5 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

(*Side comment*: *the 5 vectors used to describe* $Col(A)$ *cannot be linearly independent in this example: why ? In fact, removing redundant vectors, we can write the column space as*

$$\text{Col}(A) = \textit{Span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

but don't worry about that. For now, it's just an observation,)

$$A = \begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix}$$

Is some particular vector in Col(A) ?
Takes a little work:

Is $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{3} \end{bmatrix}$ **in Col(A) ?** Same as asking whether

$$\begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \text{ has a solution.}$$

$$\left[\begin{array}{ccccc|c} 5 & 10 & 2 & 9 & 4 & 1 \\ 3 & 6 & 1 & 5 & 1 & 0 \\ 2 & 4 & 1 & 4 & 2 & 3 \end{array}\right] \ldots \sim \begin{bmatrix} \mathbf{1} & 2 & 0 & 1 & 0 & -5 \\ 0 & 0 & \mathbf{1} & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & \mathbf{1} & -2 \end{bmatrix}$$

Yes: the system is consistent (and there are free variables, so there are actually infinitely many solutions).

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 0 \\ 17 \\ 0 \\ -2 \end{bmatrix} + \boldsymbol{x_2} \begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{x_4} \begin{bmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{-2} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \text{ so, for example,}$$

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 0 \\ 17 \\ 0 \\ 2 \end{bmatrix} \text{ works :}$$

$$T\left(\begin{bmatrix} -5 \\ 0 \\ 17 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 & 10 & 2 & 9 & 4 \\ 3 & 6 & 1 & 5 & 1 \\ 2 & 4 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 17 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{3} \end{bmatrix}$$