Review from last time

A vector space V is a nonempty set of objects (which we will call vectors) and two operations which we call

i) "addition" $(+)$
ii) "multiplication by a scalar" such that:

For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ (vectors)
and for all $c, d \in \mathbb{R}$ (scalars)

1. $u + v$ is in V (closure under addition)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector (denoted 0) with the property $u + 0 = u$
5. For each vector u, there is another vector called $-u$ for which $u + (-u) = 0$

6. $c\boldsymbol{u}$ is in V (closure under scalar multiplication)
7. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
10. $1\boldsymbol{u} = \boldsymbol{u}$

Suppose $V =$ a vector space
and that H = a subset of V (written $H \subseteq V$)

H is called a subspace of V if

i) $\mathbf{0} \in H$

ii) if $\boldsymbol{u}, \boldsymbol{v} \in H$, then $\boldsymbol{u} + \boldsymbol{v} \in H$
(H is "closed under addition")

iii) if $\boldsymbol{u} \in H$ and c is a scalar, then $c\boldsymbol{u} \in H$
(H is "closed under scalar multiplication")

All other properties needed for a vector space are satisfied by vectors in H automatically since they are true already in the given vector space V:

for example, if $\boldsymbol{u}, \boldsymbol{v} \in H$, then $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
because $\boldsymbol{u}, \boldsymbol{v} \in V$ we already know
$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ is true in V.

So you can think of a subspace H as a smaller "self-contained" vector space that "lives inside" the larger vector space V.

Theorem If $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}$ are vectors in V, then

Let $H = \text{span}\left\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}\right\}$
$= \left\{\text{all possible linear combinations of } \boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}\right\}$

H* is a subspace of *V

Definition Suppose $T : V \to W$, where V, W are vector spaces,
T is called a linear transformation if

i) for all $\boldsymbol{x}$, $\boldsymbol{y}$ in V, $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
ii) for all $\boldsymbol{x}$ in V and all scalars c, $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

This definition looks exactlt the same as when $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$.
But for arbitrary vector spaces V, W we cannot find a matrix at this time to represent T – because we have no coordinates and we used coordinates of vectors to construct matrices for linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$.

(*We will see a way later that we san sometimes reintroduce matrices into this more general situation.*)

Practice

1) Let $\mathbb{P}$ be the vector space of all polynomials

a) What is the "zero vector" in this space?

b) Is the "zero vector" in Span $\{1, t, t^2, t^3\}$?
Why ?

c) Consider a linear combination of these "vectors"

$$1, t, t^2, t^3$$

that adds up to the zero vector:

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 = \mathbf{0}$$

What can you say about the weights $c_0, ..., c_3$?
Why?

2) Consider two vector spaces:

$\mathcal{C} = \{f : f$ is a continuous function with domain $(-\infty, \infty)\}$

$\mathcal{D} = \{f : f$ is a differentiable function with domain $(-\infty, \infty)$

a) What is an example of a subspace of $\mathcal{D}$?

b) i) Can you name a function f that is in one of these vector spaces but not in the other?

ii) Which one is true?

$\mathcal{C}$ is a subspace of $\mathcal{D}$ or $\mathcal{D}$ is a subspace of $\mathcal{C}$

c) Can you think of a linear transformation

$$T : \mathcal{D} \to \mathcal{C} \ ?$$

Can you think of a second one ?